mathematics

Article

Theorems for Boyd–Wong Contraction Mappings on Similarity Spaces

Ondrej Rozinek [1,*] and Monika Borkovcova [2]

1 Department of Process Control, University of Pardubice, 532 10 Pardubice, Czech Republic
2 Department of Information Technology, University of Pardubice, 532 10 Pardubice, Czech Republic; monika.borkovcova@upce.cz
* Correspondence: ondrej.rozinek@gmail.com

Abstract: In this article, we introduce novel fixed point results for Boyd–Wong-type contraction mappings within the framework of similarity spaces, which have broad practical applications. The development of these results extends the classical theory of Boyd–Wong contractions by exploiting the unique structure and properties of similarity spaces. We provide an in-depth examination of the derived contractions, establishing conditions under which fixed points exist and are unique. In the latter part of the paper, we illustrate the applicability and effectiveness of the proposed results with representative examples.

Keywords: similarity space; Boyd–Wong contraction; similarity contraction

MSC: 37C25

check for updates

Citation: Rozinek, O.; Borkovcova, M. Theorems for Boyd–Wong Contraction Mappings on Similarity Spaces. *Mathematics* **2023**, *11*, 4359. https://doi.org/10.3390/math11204359

Academic Editor: Óscar Valero Sierra

Received: 21 September 2023
Revised: 16 October 2023
Accepted: 18 October 2023
Published: 20 October 2023

1. Introduction

Similarity and dissimilarity functions are essential tools in numerous research fields, including information retrieval, data mining, machine learning, cluster analysis, and various applications in database searches and protein sequence comparisons. The use of dissimilarity functions typically necessitates a metric space, which is a well-defined mathematical structure. However, the concept of similarity functions lacks a formally accepted definition, resulting in ambiguity and inconsistency in their utilization. To address this gap, we aim to establish a viable theory of similarity space by constructing it as a duality to metric space [1].

2. Preliminaries

Banach fixed-point theorem [2], a fundamental cornerstone of metric space theory, serves as a powerful tool for a myriad of analytical problems. The theorem has been extensively studied, emphasizing its diverse applications in analysis. The Banach fixed-point theorem not only asserts the existence and uniqueness of fixed points for certain self-mappings in metric spaces, but also provides a constructive method for their discovery, thus endowing it with significant practical utility.

This theorem states that, if X is a non-empty set and d is a metric on X such that $d\colon \to X \times X \to \mathbb{R}^+$ and if T is a self-mapping on a complete metric space (X, d), which satisfies

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1}$$

for some $\alpha \in (0, 1)$ for all x, y, then T has a unique fixed point x^* and a sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^* for $x \in X$.

One of the main generalizations of the Banach principle is the theorem proposed by D.W. Boyd and J.S. Wong in [3].

We denote the range of d by P, and the closure of P by $\overline{P}$, so $P = \{d(x, y) \mid x, y \in X\}$. This is also applicable within the context of a similarity space.

Theorem 1 (Boyd–Wong Theorem)**.** *Let (X,d) be a complete metric space with P defined as $d(x,y)\colon x,y\in X$. Let $T\colon X\to X$ be a self-mapping satisfying:*

$$d(Tx,Ty)\leq\psi(d(x,y)) \tag{2}$$

for each $x,y\in X$ where $\psi\colon\overline{P}\to\mathbb{R}^+$ is upper semicontinuous from the right on $\overline{P}$ and satisfies $\psi(t)<t$ for all $t\in\overline{P}\setminus\{0\}$, where $\overline{P}$ denotes the closure P. Then, T has a unique fixed point x^* and $d(T^nx,x^*)\to 0$ for each $x\in X$.

The Boyd–Wong theorem's applicability has been extensively studied across various abstract mathematical spaces. Notably, its principles have been examined in the context of partially ordered metric spaces [4], cone metric spaces [5], and generalized metric spaces [6]. Further investigations have been carried out in partial metric spaces [7–9], quasi-metric spaces [10], b-metric spaces [11], and bipolar metric spaces [12]. Other applications and generalizations can be found in [13,14].

3. Similarity Space

Recent advancements have extended its influence to the realm of similarity spaces as well. Recall the definition.

Definition 1 (Similarity Space [1,15–17])**.** *A similarity on a nonempty set X is a function $s\colon X\times X\to\mathbb{R}^+$ such that for all elements $x,y,z\in X$:*

(S1) $s(x,y)=s(y,x)$ (symmetry),
(S2) $s(x,z)+s(y,y)\geq s(x,y)+s(y,z)$ (triangle inequality),
(S3) $s(x,x)=s(x,y)=s(y,y)\iff x=y$ (identity of indiscernibles),
(S4) $s(x,y)\geq 0$ (non-negativity),
(S5) $s(x,y)\leq\min\{s(x,x),s(y,y)\}$ (bounded self-similarity).

A similarity space is an ordered pair (X,s) such that X is a nonempty set and s is a similarity on X.

A few issues require attention. The name 'similarity metric' is a convention already suggested in the preceding. Calling it a 'metric' should be understood in the sense of a monotonously decreasing convex transformation of a partial metric or a distance metric [1]. In this paper, we use only the term 'similarity' to avoid any misunderstanding. We note that this definition allows for positive self-similarity, $s(x,x)>0$, and different self-similarities, $s(x,x)\neq s(y,y)$. However, if $x=y$, then $s(x,y)$ may not necessarily be 0. The theory of similarity space is very close to the theories of metric spaces and partial metrics, and some parts of this paper were also inspired by these theories [18–20].

A basic example of a similarity space is the ordered pair $(\mathbb{R}^+,s)$, defined as follows:

$$s(x,y)=x\cap y=\min\{x,y\}=\frac{x+y-|x-y|}{2} \tag{3}$$

for all x,y in $\mathbb{R}^+$. Other examples of similarity spaces that are interesting in terms of broad practical applications, such as Jaccard index, Tanimoto coefficient, Generalized Rozinek similarity, Levenshtein similarity, and longest common subsequence, can be found in [1].

Definition 2. *Let (X,s) be a similarity space. Then,*

(i) A sequence $\{x_n\}_{n=1}^\infty$ in a similarity space (X,s) converges to an element $x\in X$ if and only if

$$\lim_{n\to\infty}s(x_n,x)=s(x,x). \tag{4}$$

(ii) A sequence $\{x_n\}_{n=1}^\infty$ in a similarity space (X,s) is called a Cauchy sequence if there exists

$$\lim_{n,m\to\infty}s(x_n,x_m)=s(x,x). \tag{5}$$

(iii) *A similarity space* (X, s) *is said to be complete if every Cauchy sequence* $\{x_n\}_{n=1}^{\infty}$ *with respect to an element* $x \in X$ *has a limit* $\lim_{n,m\to\infty} s(x_n, x_m) = s(x, x)$ *that is also in* X.

Theorem 2 (Induced Elementary Metric). *Let* $x, y \in X$. *If* $s(x, y)$ *is a similarity on* X, *then the function* $d^s\colon X \times X \to \mathbb{R}^+$ *given by*

$$d^s(x, y) = s(x, x) + s(y, y) - 2s(x, y) \tag{6}$$

is an induced elementary metric on X.

Proof. Consider $x,\ y \in X$. Then, $d^s(x, y) = s(x, x) + s(y, y) - 2s(x, y)$ is always non-negative due to the bounded self-similarity (S5), since $s(x, y) \leq \min s(x, x), s(y, y)$. Moreover, if $d^s(x, y) = d^s(y, x) = 0$, we obtain $x = y$ because $s(x, x) = s(x, y) = s(y, y)$. Furthermore, the triangular inequality holds

$$\begin{aligned} d^s(x, y) &= s(x, x) + s(y, y) - 2s(x, y) \\ &\leq s(x, x) + s(y, y) - 2[s(x, z) + s(y, z) - s(z, z)] \\ &= [s(x, x) + s(z, z) - 2s(x, z)] + [s(y, y) + s(z, z) - 2s(y, z))] \\ &= d^s(x, z) + d^s(y, z). \end{aligned}$$

□

Definition 3 (Open s-Ball and Closed s-Ball). *Let* (X, s) *be the similarity space, and let* $x \in X$ *and* $\epsilon \geq 0$. *The open s-ball of radius* ϵ *with center* x *is the set*

$$B_s(x, \epsilon) = \{y \in X\colon s(x, y) > \min\{s(x, x), s(y, y)\} - \epsilon\}. \tag{7}$$

The closed s-ball of radius ϵ *with center* x *is the set*

$$\overline{B}_s(x, \epsilon) = \{y \in X\colon s(x, y) \geq \min\{s(x, x), s(y, y)\} - \epsilon\}. \tag{8}$$

4. Main Results

The main contribution is the study of the dual relationship between the similarity space and the metric space, where the similarity space forms a different axiomatic system. We focus on the dualistic view of Boyd–Wong contraction and then purely from the perspective of similarity spaces. We demonstrate our derivations through several examples.

Theorem 3 (Boyd-Wong Dualistic Contraction). *Let* (X, s) *be a complete similarity space and so its dual complete metric space* (X, d^s) *and let* $T\colon X \to X$ *satisfy the following condition:*

$$\begin{aligned} &s(Tx, Tx) + s(Ty, Ty) - 2s(Tx, Ty) \\ &\leq \varphi(s(x, x)) + \varphi(s(y, y)) - 2\psi(s(x, y)), \quad \forall x, y \in X \end{aligned} \tag{9}$$

where $\varphi\colon \overline{P} \to \mathbb{R}^+$ *is upper semicontinuous from the right on* $\overline{P}$ *and* $\varphi(t) < t$ *for all* $t \in \overline{P} \setminus \{0\}$, *respectively,* $\psi\colon \overline{P} \to \mathbb{R}^+$ *is lower semicontinuous from the right on* $\overline{P}$ *and* $\psi(t) > t$ *for all* $t \in \overline{P} \setminus \{0\}$. *Then,* T *has a unique fixed point and every sequence* $\{T^n x\}_{n\in\mathbb{N}}$ *converges to this unique fixed point* x^*.

Proof. Let $x, y \in X$, since the complete similarity space implies a complete metric space, we may assume

$$d^s(x, y) \leq \varphi(d^s(x, y)). \tag{10}$$

Furthermore, we express from the previous inequality

$$\begin{aligned} &s(Tx,Tx) + s(Ty,Ty) - 2s(Tx,Ty) \\ &\leq \varphi(s(x,x) + s(y,y) - 2s(x,y)) \\ &\leq \varphi(s(x,x)) + \varphi(s(y,y)) + 2\varphi(-s(x,y)) \\ &= \varphi(s(x,x)) + \varphi(s(y,y)) - 2\psi(s(x,y)). \end{aligned} \tag{11}$$

So the claim is proven. □

In similarity spaces, defining contraction involves a unique duality, represented by self-similarity and mutual similarity conditions. This stands in contrast to traditional metric space contractions. The proposed definition establishes a bifurcated condition for contraction—self-similarity and mutual similarity—ensuring that both intrinsic and comparative similarity are bounded and well-regulated. This is crucial because in similarity spaces, ensuring just intrinsic similarity does not guarantee consistent contraction throughout the entire similarity space.

Theorem 4 (Boyd-Wong Similarity Contraction)**.** *Let X be a complete similarity space, and let $T\colon X \to X$ satisfy the contraction conditions of self-similarity*

$$s(Tx,Tx) \leq \varphi(s(x,x)), \quad \forall x \in X \tag{12}$$

and mutual similarity

$$s(Tx,Ty) \geq \psi(s(x,y)), \quad \forall x,y \in X \tag{13}$$

where $\varphi\colon \overline{P} \to \mathbb{R}^+$ is upper semicontinuous from the right on $\overline{P}$ and $\varphi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$, respectively, $\psi\colon \overline{P} \to \mathbb{R}^+$ is lower semicontinuous from the right on $\overline{P}$ and $\psi(t) > t$ for all $t \in \overline{P} \setminus \{0\}$. Then,

(i) T has a unique fixed point $x^* \in X$,
(ii) for every $x \in X$, the Picard sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*:

$$\lim_{n\to\infty} T^n x = x^*. \tag{14}$$

Proof. Let $x \in X$, define a shorter notation for self-similarity and mutual similarity

$$u_n = s(T^n x, T^n x) \qquad \text{(self-similarity)}, \tag{15}$$

$$w_n = s(T^n x, T^{n-1} x) \qquad \text{(mutual similarity)}. \tag{16}$$

The sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are monotonically decreasing and increasing, respectively. Since both sequences are bounded, they are convergent. Let us denote the limits of these sequences as $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} w_n = w$. To ensure the conditions of the theorem are satisfied, we need to show that, $u_n, w_n \to 0$ as $n \to \infty$. But, if $u, w > 0$, we have

$$\begin{aligned} u_{n+1} &\leq \varphi(u_n), \\ w_{n+1} &\geq \psi(w_n). \end{aligned} \tag{17}$$

So that

$$\begin{aligned} u &= \lim_{n\to\infty} u_n = \limsup_{n\to\infty} u_n \leq \limsup_{t\to u_+} \varphi(t) \leq \varphi(u), \\ w &= \lim_{n\to\infty} w_n = \liminf_{n\to\infty} w_n \geq \liminf_{t\to w_+} \psi(t) \geq \psi(w), \end{aligned} \tag{18}$$

which is a contradiction because $\varphi(u) \not< u$ and $\psi(w) \not> w$ as in the statement. Thus, u_n, w_n converges to zero as $n \to \infty$ for each $x \in X$.

We now show that $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence for each $x \in X$. This will complete the proof, since the limit of this sequence is a fixed point x^* of T which is clearly unique. Suppose that $\{T^n x\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Then, for some $\epsilon, \delta > 0$ and each $k \in \mathbb{N}$, we can find sequences of natural numbers $\{m(k)\}_{n\in\mathbb{N}}$ and $\{n(k)\}_{n\in\mathbb{N}}$ with $m(k) > n(k) \geq k$ such that for all $k \in \mathbb{N}$ and from Definition 3, we apply the closed s-ball and its complement in this manner:

$$\begin{aligned} s_k = s(T^{m(k)}x, T^{n(k)}x) &\leq \delta \\ &= \min\{s(T^{m(k)}x, T^{m(k)}x), s(T^{n(k)}x, T^{n(k)}x)\} - \epsilon \end{aligned} \tag{19}$$

And

$$\begin{aligned} s(T^{m(k)-1}x, T^{n(k)}x) &> \delta \\ &= \min\{s(T^{m(k)-1}x, T^{m(k)-1}x), s(T^{n(k)}x, T^{n(k)}x)\} - \epsilon. \end{aligned} \tag{20}$$

This can be accomplished by choosing $m(k)$ as the least natural number exceeding $n(k)$ for satisfying the above inequality. Now,

$$\begin{aligned} s_k &= s(T^{m(k)}x, T^{n(k)}x) \\ &\geq s(T^{m(k)}x, T^{m(k)-1}x) + s(T^{m(k)-1}x, T^{n(k)}x) - s(T^{m(k)-1}x, T^{m(k)-1}x) \\ &\geq w_{m(k)} + \delta - u_{m(k)}. \end{aligned} \tag{21}$$

Thus, $\delta \geq s_k \geq w_{m(k)} + \delta - u_{m(k)}$. Consequently,

$$\begin{aligned} \delta &\geq \limsup_{k\to\infty} s_k \geq \liminf_{k\to\infty} s_k \geq \liminf_{k\to\infty}(w_{m(k)} + \delta - u_{m(k)}) \\ &= \liminf_{k\to\infty} w_{m(k)} + \delta - \limsup_{k\to\infty} u_{m(k)} \\ &= \delta. \end{aligned} \tag{22}$$

Hence, $\lim_{k\to\infty} s_k = \delta$. Indeed, $s_k \to \delta^+$ as $k \to \infty$.
Further,

$$\begin{aligned} s_k &= s(T^{m(k)}x, T^{n(k)}x) \\ &\geq s(T^{m(k)}x, T^{m(k)+1}x) + s(T^{m(k)+1}x, T^{n(k)+1}x) + s(T^{n(k)+1}x, T^{n(k)}x) \\ &\quad - s(T^{m(k)+1}x, T^{m(k)+1}x) - s(T^{n(k)+1}x, T^{n(k)+1}x) \\ &\geq w_{m(k)} + \psi(s(T^{m(k)}x, T^{n(k)}x) + w_{n(k)} - u_{m(k)+1} - u_{n(k)+1} \\ &\geq 2w_k + \psi(s_k) - 2u_k. \end{aligned} \tag{23}$$

Taking the limit as $k \to +\infty$ in the above inequality, it follows that

$$\liminf_{k\to\infty} s_k = \lim_{k\to\infty} s_k = \delta^+ \geq \liminf_{k\to\infty} \psi(s_k) \geq \psi(\delta). \tag{24}$$

Since $\delta > 0$, this contradicts that $\delta > \psi(\delta)$. Hence, $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. As X is complete, it converges to an element x^* in X. Since for all x, $y \in X$, $s(Tx, Ty) \geq \psi(s(x, y)$ and $s(Tx, Tx) \leq \varphi(s(x, x))$, T is continuous. Since $\{T^{n+1}x\}_{n\in\mathbb{N}}$ converges to Tx^* and is also a subsequence of $\{T^n x\}_{n\in\mathbb{N}}$, it follows that $x^* = Tx^*$. Since $\varphi(t) < t$ and $\psi(t) > t$ for all $t > 0$, it follows that the fixed point x^* of T is unique. □

5. Applications

Example 1. *Let (X, s) be a similarity space such that $X = [0, 1] \subset \mathbb{R}$ and $s\colon X \times X \to \mathbb{R}^+$ be defined by*

$$s(x,y) = x \cap y = \min\{x,y\} = \frac{x+y-|x-y|}{2} \tag{25}$$

for all $x, y \in X$. Clearly, (X,s) is a complete similarity space. Define the operator T associated with the quadratic function $T\colon X \times X$ by $Tx = x - \frac{1}{2}x^2 + 1$. Without loss of generalization, we can assume that $x > y$, $\min\{x,y\} = t$, then

$$\begin{aligned} s(Tx,Ty) &= \frac{x - \frac{1}{2}x^2 + 1 + y - \frac{1}{2}y^2 + 1 - (x - \frac{1}{2}x^2 + 1 - (y - \frac{1}{2}y^2 + 1))}{2} \\ &= \frac{2y - y^2 + 2}{2} = y - \frac{1}{2}y^2 + 1 \end{aligned} \tag{26}$$

and

$$s(Tx,Tx) = x - \frac{1}{2}x^2 + 1. \tag{27}$$

Thus, if we define

$$\varphi(t) \le t - \frac{1}{2}t^2 + 1, \quad \forall x \in X \qquad \textit{(self-similarity)} \tag{28}$$

$$\psi(t) \ge t + \frac{1}{2}t^2 + 1, \quad \forall x, y \in X \qquad \textit{(mutual-similarity)} \tag{29}$$

where $X = [0,1] \subset \mathbb{R}$. Then, φ is upper semicontinuous from the right on $\mathbb{R}^+$, $\varphi(t) < t$ for all $t > 0$, respectively, ψ is lower semicontinuous from the right on $\mathbb{R}^+$, $\psi(t) > t$ for all $t > 0$.

Example 2 (Three-Point Boundary Value Problem [14])**.** *We investigated the existence of at least one solution for the second-order differential equations:*

$$\begin{cases} u'' = f(t,u,u'), & 0 < t < 1, \\ u(0) = 0, u(t_0) = g(u(\eta)), \end{cases} \tag{30}$$

where $0 < \eta < t_0 < 1$, $f(t)$, $g(t)$ are non-negative continuous functions and $u \in C^1[0,\ t_0]$.

The problem can be written equivalently in integral form as

$$u(t) = \int_0^{t_0} G(t,s) f(s,u(s),u'(s)\, ds + \frac{t}{t_0} g(u(\eta)), \tag{31}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} -\frac{t(t_0-s)}{t_0}, & 0 \le t \le s \le t_0, \\ -\frac{s(t_0-t)}{t_0}, & 0 \le s \le t \le t_0. \end{cases} \tag{32}$$

Furthermore,

$$\max_{C^1[0,t_0]} \int_0^{t_0} |G(t,s)|\, ds = \max_{C^1[0,t_0]} \frac{t(t_0-t)}{2} = \frac{t_0^2}{8}. \tag{33}$$

We observe that u is a solution if and only if u is a fixed point of the operator $T\colon C^1[0,t_0] \to C^1[0,t_0]$, defined by

$$(Tu)(t) = \int_0^{t_0} G(t,s) f(s,u(s),u'(s)\, ds + \frac{t}{t_0} g(u(\eta)), \tag{34}$$

where $C^1[0,t_0]$ denotes the space of all continuously differentiable functions defined on $[0,t_0]$. Consider the similarity space of continuously differentiable functions $C^1[0,t_0]$ equipped with the similarity s_∞ defined as

$$s_\infty(f,g) = f(t) \cap f(t) = \inf_{t \in [0,t_0]} \{f(t), g(t)\} \tag{35}$$

where $f(t), g(t) \in C^1[a,b]$ are any non-negative real functions. There exist $\varphi_1(t), \varphi_2(t) \colon \mathbb{R}^+ \to \mathbb{R}^+$ upper semicontinous from the right and $\psi_1(t), \psi_2(t) \colon \mathbb{R}^+$ lower semicontinous from the right such that

$$\begin{cases} f(t,u,v) \leq \varphi_1(u) + \varphi_2(v) \\ g(u) \leq c \cdot u \\ f(t,\overline{u},\overline{v}) \leq \varphi_1(\overline{u}) + \varphi_2(\overline{v}) \\ g(\overline{u}) \leq c \cdot \overline{u} \end{cases} \quad \text{(self-similarity)} \tag{36}$$

and

$$\begin{cases} s_\infty(f(t,u,v), f(t,\overline{u},\overline{v}) \geq \psi_1(s_\infty(u,\overline{u})) + \psi_2(s_\infty(v,\overline{v})) \\ s_\infty(g(u), g(\overline{u}) \geq c \cdot s_\infty(u,\overline{u}) \end{cases} \quad \text{(mutual-similarity)} \tag{37}$$

for $t \in [0, t_0]$ and u, $\overline{u}$, v, $\overline{v} \in \mathbb{R}^+$ and $c > 0$. If $f(t)$, $g(t)$ satisfies the condition previous and

$$\varphi(t) := \max\left\{ \frac{t_0^2}{8}(\varphi_1 + \varphi_2)(t) + c \cdot t, \frac{t_0}{2}(\varphi_1 + \varphi_2)(t) + \frac{c}{t_0}t \right\} < t \quad \text{(self-similarity)} \tag{38}$$

$$\psi(t) := \min\left\{ \frac{t_0^2}{8}(\psi_1 + \psi_2)(t) + c \cdot t, \frac{t_0}{2}(\psi_1 + \psi_2)(t) + \frac{c}{t_0}t \right\} > t \quad \text{(mutual-similarity)} \tag{39}$$

Then, the three-point boundary value problem has a unique solution. Moreover, this solution can be obtained as a limit of the sequence of successive approximations.

We now prove this statement for self-similarity:

$$s_\infty(T(u)(t), T(u)(t)) = T(u)(t) \tag{40}$$

$$= \int_0^{t_0} G(t,s) f(s, u(s), u'(s)\, ds + \frac{t}{t_0} g(u(\eta)) \tag{41}$$

$$\leq \int_0^{t_0} |G(t,s)| (\varphi_1(u(s)) + \varphi_2(u'(s))\, ds + \frac{t}{t_0} \cdot c \cdot u(\eta) \tag{42}$$

$$\leq (\varphi_1(u(s)) + \varphi_2(u'(s)) \int_0^{t_0} |G(t,s)|\, ds + \frac{t}{t_0} \cdot c \cdot u(\eta) \tag{43}$$

$$\leq \frac{t_0^2}{8}(\varphi_1 + \varphi_2) u(t) + c \cdot u(t). \tag{44}$$

Analogously, we obtain

$$s_\infty(T(u)'(t), T(u)'(t)) = T(u)'(t) \leq \frac{t_0}{2}(\varphi_1 + \varphi_2) u'(t) + c \cdot u'(t). \tag{45}$$

We will now show that the statement also holds for mutual similarity

$$
\begin{aligned}
s_\infty(T(u)(t), T(\overline{u}(t)) &= \inf_{t\in[0,t_0]}\{T(u)(t), T(\overline{u})(t)\} \\
&= \inf_{t\in[0,t_0]}\left\{ \int_0^{t_0} G(t,s)f(s,u(s),u'(s)\,ds + \frac{t}{t_0}g(u(\eta)), \right. \\
&\qquad \left. \int_0^{t_0} G(t,s)f(s,\overline{u}(s),\overline{u}'(s)\,ds + \frac{t}{t_0}g(\overline{u}(\eta)) \right\} \\
&\geq \inf_{t\in[0,t_0]}\left\{ \int_0^{t_0} G(t,s)f(s,u(s),u'(s)\,ds, \int_0^{t_0} G(t,s)f(s,\overline{u}(s),\overline{u}'(s)\,ds \right\} \\
&\quad + \inf_{t\in[0,t_0]}\left\{ \frac{t}{t_0}g(u(\eta)), \frac{t}{t_0}g(\overline{u}(\eta)) \right\} \\
&\geq \int_0^{t_0} |G(t,s)|[\psi_1(\inf_{t\in[0,t_0]}\{u(s),\overline{u}(s)\}) + \psi_2(\inf_{t\in[0,t_0]}\{u'(s),\overline{u}'(s)\})]\,ds \\
&\quad + \frac{t}{t_0}\cdot c \cdot \inf_{t\in[0,t_0]}\{u(\eta),\overline{u}(\eta)\} \\
&\geq [\psi_1(\inf_{t\in[0,t_0]}\{u(s),\overline{u}(s)\}) + \psi_2(\inf_{t\in[0,t_0]}\{u'(s),\overline{u}'(s)\})] \int_0^{t_0} |G(t,s)|\,ds \\
&\quad + c \cdot \inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)\} \\
&\geq \frac{t_0^2}{8}(\psi_1+\psi_2)(\inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)s\}) + c\cdot \inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)\}.
\end{aligned}
$$

Similarly, we obtain

$$
s_\infty(T(u)'(t), T(\overline{u})'(t)) \geq \frac{t_0}{2}(\psi_1+\psi_2)(\inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)s\}) + \frac{c}{t_0}\cdot \inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)\}. \tag{46}
$$

Combining the results for both self-similarity and mutual-similarity, we find that

$$
s_\infty(T(u)(t), T(u)(t)) \tag{47}
$$

$$
\leq \max\left\{ \frac{t_0^2}{8}(\varphi_1+\varphi_2)u(t) + c\cdot u(t), \frac{t_0}{2}(\varphi_1+\varphi_2)u(t) + \frac{c}{t_0}u(t) \right\} < t \tag{48}
$$

$$
s_\infty(T(u)(t), T(u)(t)), \tag{49}
$$

$$
\geq \min\left\{ \frac{t_0^2}{8}(\psi_1+\psi_2)(\inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)s\}) + c\cdot \inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)\}, \right. \tag{50}
$$

$$
\left. \frac{t_0}{2}(\psi_1+\psi_2)(\inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)s\}) + \frac{c}{t_0}\cdot \inf_{t\in[0,t_0]}\{u(t),\overline{u}(t)\} \right\} > t. \tag{51}
$$

Since the $\varphi(t)$, $\psi(t)$ conditions hold for all $t > 0$, then Boyd–Wong Similarity Contraction (Theorem 4) can be applied and T has a unique fixed point.

6. Discussion

In our study, a notable observation pertains to bifurcation in similarity behaviors. Specifically, even small changes can lead to a significant shift in behavior, transitioning from mutual similarity to self-similarity. Such a bifurcated approach offers a nuanced perspective on understanding the dynamics of similarity spaces.

In theories like fractal theory and Hausdorff dimension, the primary emphasis is on characterizing metric spaces rather than directly addressing the intrinsic properties of spaces from a similarity viewpoint. There exists no established axiomatic system for similarity in these contexts. This highlights the motivation for introducing similarity spaces, where notions such as self-similarity are inherently defined.

To the best of our knowledge, we are the first authors to delve into this unique bifurcated structure and its implications. We find this discovery particularly promising, and we believe that this theory has the potential to shed light on some of the unexplained and unsolved open problems in the field. Further exploration and applications of this bifurcation phenomenon in various domains could pave the way for further advancements.

7. Conclusions

For the first time, we investigate the existence and uniqueness of fixed points in the newly created similarity space, which itself forms a different axiomatic system than metric space. This space encompasses general properties of a class of similarity functions with broad applicability. Central to our exploration was the fixed-point theory, ranging from the foundational Banach fixed-point theorem to the nuanced generalization introduced by the Boyd–Wong theorem.

Significantly, we introduced bifurcated conditions for both self-similarity and mutual similarity. These conditions were instrumental in demonstrating the existence of solutions for three boundary value problems related to second-order differential equations.

The establishment of a similarity space, constructed as a duality to metric space, adds depth to our understanding of these functions. The fixed-point results, coupled with the newly introduced conditions, hold considerable promise for further theoretical developments and practical applications.

Author Contributions: Conceptualization, O.R. and M.B.; methodology, O.R.; formal analysis, O.R.; writing—original draft preparation, O.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by an SGS grant from the Faculty of Electrical Engineering and Informatics, University of Pardubice, Czech Republic. This support is very gratefully acknowledged.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rozinek, O.; Mareš, J. The Duality of Similarity and Metric Spaces. *Appl. Sci.* **2021**, *11*, 1910. [CrossRef]
2. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
3. Boyd, D.W.; Wong, J.S. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
4. Aydi, H.; Shatanawi, W.; Postolache, M.; Mustafa, Z.; Tahat, N. Theorems for Boyd-Wong-type contractions in ordered metric spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 359054. [CrossRef]
5. Arandjelović, I.; Kadelburg, Z.; Radenović, S. Boyd–Wong-type common fixed point results in cone metric spaces. *Appl. Math. Comput.* **2011**, *217*, 7167–7171. [CrossRef]
6. Kadelburg, Z.; Radenovic, S.; Shukla, S. Boyd-Wong and Meir-Keeler type theorems in generalized metric spaces. *J. Adv. Math. Stud.* **2016**, *9*, 83–93.
7. Romaguera, S. Fixed point theorems for generalized contractions on partial metric spaces. *Topol. Appl.* **2012**, *159*, 194–199. [CrossRef]
8. Nziku, F.; Kumar, S. Boyd and Wong Type Fixed Point Theorems in Partial Metric Spaces. *Moroc. J. Pure Appl. Anal.* **2019**, *5*, 251–262. [CrossRef]
9. Hussain, N.; Kadelburg, Z.; Radenović, S.; Al-Solamy, F. Comparison functions and fixed point results in partial metric spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 605781. [CrossRef]
10. Romaguera, S.; Tirado, P. The Meir–Keeler fixed point theorem for quasi-metric spaces and some consequences. *Symmetry* **2019**, *11*, 741. [CrossRef]
11. Castillo, R.E.; Morales, J.R.; Rojas, E.M. Some Boyd–Wong contraction type mappings in b-metric spaces. *J. Anal.* **2023**, *31*, 911–944. [CrossRef]
12. Murthy, P.P.; Mitrovic, Z.; Dhuri, C.P.; Radenovic, S. The common fixed points in a bipolar metric space. *Gulf J. Math.* **2022**, *12*, 31–38. [CrossRef]
13. Singh, D.; Chauhan, V.; Kumam, P.; Joshi, V.; Thounthong, P. Applications of fixed point results for cyclic Boyd–Wong type generalized F–ψ-contractions to dynamic programming. *J. Math. Comput. Sci.* **2017**, *17*, 200–215. [CrossRef]
14. Nica, O. Existence results for second order three-point boundary value problems. *Differ. Equ. Appl.* **2012**, *4*, 547–570. [CrossRef]

15. Ma, B.; Zhang, K. The similarity metric and the distance metric. In Proceedings of the 6th Atlantic Symposium on Computational Biology and GenomeInformatics, Salt Lake City, UT, USA, July 2005; pp. 1239–1242.
16. Elzinga, C.C.H.; Studer, M.M. Normalization of Distance and Similarity in Sequence Analysis. In Proceedings of the Sequence Analysis and Related Methods (LaCOSA II), Lausanne, Switzerland, 8–10 June 2016; p. 445.
17. Alhajjar, E.; Lefèvre, C. On The Similarity Metric. *Math. Mil.* **2019**, *24*, 4.
18. Matthews, S.G. *Partial Metric Spaces*; Technical Report; University of Warwick, Department of Computer Science: Coventry, UK, 1992; Unpublished.
19. Matthews, S.G. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
20. Oltra, S.; Valero, O. Banach's fixed point theorem for partial metric spaces. *Rend. Istid. Math. Univ. Trieste* **2004**, *36*, 17–26.