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Verification and estimation of uncertainties of Tobias Mayer's 18th century astronomical observations

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A B S T R A C T

Perceiving the uncertainty of the measurement has been changing over the past centuries, reflecting the advancement in the experimental techniques, the urge for reliable and reproducible measurement methodology, and development of mathematical data processing and evaluating algorithms. From the historical perspective, the concepts of considering the measurement uncertainty were firstly introduced with geographic and cartographic measurements. In this context, the works of Tobias Mayer on lunar landscape measurements are widely highlighted which, at that time, presented innovative approaches in data processing with the method of averages and pioneeringly addressed the issue of measurement error. In this study, we analyze in details the Mayer's set of 27 non-linear equations with 3 unknown parameters and discuss the effect of Mayer's linearization and subsequent mathematical procedures on the accuracy of the parameter values in contrast with the results from rigorous treatment of non-linear regression model involving the least-square method. In particular, we compare the values of the unknown parameters and their uncertainties in several variants in the linearized and non-linearized model, providing monitoring of a small deviation of the Mayer's linearization. The results, presented here, show that despite the conceptual and computational simplification of the Mayer's method, such an approach to data processing can be exploited, with an acceptable level of accuracy, in several practical situations even today.

Introduction

Measurement is defined as a quantitative process of finding a correct value of geometrical, physical, chemical, biological or informatics quantity, employing various experimental techniques and data processing routines [1]. Measurement has been a crucial part of human civilization from its beginnings and has evolved to a sophisticated scientific discipline, providing a methodology to assess parameters of materials, phenomena, and processes in physical, chemical, engineering and informatics fields, and quantitative investigations in social sciences and economics. With a development of measuring techniques and methods, especially in physics and technical fields, it became necessary to unify the systems of measuring units and settle the rules to transfer the units of measurement from one measuring device to another, guaranteeing a certain level of measurement uncertainty. This gave rise to a foundation of metrology [1,2] – a scientific and technical branch covering the aspects of development of measurement methods and measurements of various technical and physical quantities, aiming at finding a relationship between the measured and actual value of a quantity. In other words, the mission of metrology is to define and maintain uniform and accurate measurements in the fields of science, industry, economy and public administration.

It is well known that a value of a given quantity is measured with an uncertainty [3–8]. The measurement uncertainty is a parameter related to the measurement result itself, characterizing the range of values that can be rationally assigned to the measured quantity. The measurement uncertainty consists of several partial uncertainties (i.e., components of uncertainty). From a general perspective, the uncertainties are frequently classified in the two groups based on the method of their determination, i.e., uncertainty of type A (sometimes referred to as a statistical uncertainty) and type B (sometimes known as a systematical uncertainty) [3]. The A-type uncertainty, commonly denoted as u_A, is evaluated with any valid statistical method used for data treating (e.g., standard deviation of the mean in a series of independent observations, estimation of curve parameters used for fitting the experimental data, analysis of variance in the measurements heavily affected by random effects). On the contrary, the B-type uncertainty, commonly denoted as u_B, is estimated by non-statistical approaches considering any available

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relevant information (i.e., previous measurement data, instrument specifications from the manufacturer – instrumental measuring limits, data provided in calibration and validation reports, experience with the behaviors and properties of the relevant materials and measuring devices, uncertainties of the reference data reported in various handbooks). In the simplest case of repeated measurements, the estimate of a variable/parameter × is represented by the sample mean $\overline{x}$ and its A-type uncertainty is given by

$$u_{\mathrm{A}}(x) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2}, n \geq 10 \tag{1}$$

where n is a number of measurements, x_i are individual observations of the variable/parameter × and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. On the other hand, the calculation of the B-type uncertainty can be more complex. For example, if the maximum deviation of the j-th source of uncertainty is known from numerous observations, i.e., $z_{j,\max}$, then

$$u_{\mathrm{B},z_j} = \frac{z_{j,\max}}{k_{\mathrm{P}}} \tag{2}$$

where k_{P} is the coefficient originating from the probability distribution (i.e., normal, uniform, triangular) governing the j-th source of the uncertainty. Moreover, if u_{B,z_j} is known, for q uncorrelated sources of uncertainty, we arrive at

$$u_{\mathrm{B}} = \sqrt{\sum_{j=1}^{q} A_j^2 u_{\mathrm{B},z_j}^2}, \tag{3}$$

where u_{B,z_j} are uncertainties of individual sources and A_j represent their corresponding sensitivity coefficients.

The most common sources of uncertainty include (i) imperfect or incomplete definition of the measured quantity or its implementation, (ii) inappropriate selection of the instrument (resolution), (iii) inappropriate (unrepresentative) selection of the measurement sample, (iv) inappropriate measurement procedure, (v) simplification (rounding) of constants and assumed values, (vi) linearization, approximation, interpolation or extrapolation in the evaluation of experimental data, (vii) unknown or uncompensated environmental influences, (viii) non-compliance with identical conditions during repeated measurements, (ix) subjective influence of the operator, and (x) inaccuracy of standards and reference materials [3]. Some of them contribute to the uncertainty of type A only, some of them to the uncertainty of type B only, and some of them to both uncertainties. As the uncertainty of type A stems from statistical character of both the measurement process and nature of the physical/chemical/biological phenomena under investigation, it is always present and cannot be omitted. On the contrary, the uncertainty of type A can be, in principle, significantly reduced due a development of measurement techniques, adopting quantum nature of various quantities as measurement references. The level of uncertainty is always a matter of compromise depending on the requirement of a given field and application. Frsystematic errors of known values, the result value estimator is corrected by corrections (before the uncertainty analysis). Systematic errors with unknown values cannot be corrected in this way, and only they are expressed jointly by type B uncertainty. Type B uncertainty is therefore the standard deviation of a set of randomized systematic errors with unknown values and is obtained from their expected limits or acceptable ranges and with a predicted distribution, e.g., uniform, or normal. According to 3 definition u_{A} and u_{B} are always uncorrelated.

For systematic errors of known values, the result value estimator is corrected by corrections (before the uncertainty analysis). Systematic errors with unknown values cannot be corrected in this way, and only they are expressed jointly by type B uncertainty. Type B uncertainty is therefore the standard deviation of a set of randomized systematic errors with unknown values and is obtained from their expected limits or acceptable ranges and with a predicted distribution, e.g., uniform, or normal. According to the Guide to the expression of uncertainty in measurement [3], u_{A} and u_{B} are always uncorrelated.

In the case of direct measurements, the combined uncertainty, often denoted as u_{C}, is given as a positive square root of the sum of a square of u_{A} and u_{B} (for uncorrelated u_{A} and u_{B}). For indirect measurements and with correlations, u_{C} of the estimate y of the output variable Y (i.e., $Y = f(X_1, X_2,\ldots, X_N)$) is determined as

$$u_{\mathrm{C}}(y) = \sqrt{\sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)} \tag{4}$$

and is known as the law of propagation of uncertainty. Here, $x_1, x_2,\ldots, x_{\mathrm{N}}$ are the estimates of the input variables $X_1, X_2,\ldots, X_N$, respectively, $u(x_1)$, $u(x_2),\ldots, u(x_N)$ represent the corresponding uncertainties of the estimates $x_1, x_2,\ldots, x_N$ of the input variables $X_1, X_2,\ldots, X_N$, respectively, $\partial f/\partial x_i$, $i = 1,\ldots, N$, stands for the sensitivity coefficients, and $u(x_i, x_j)$ denotes the covariance between the estimates x_i and x_j of the input variables X_i and X_j correlated with each other. If y and $u_{\mathrm{C}}(y)$ shows approximately Gaussian (normal) distribution, then $Y = y \pm u_{\mathrm{C}}$ encompasses approximately 68 % of the distribution of the true value of the output variable Y. In other words, the true value of Y lies within the interval from $y - u_{\mathrm{C}}$ to $y + u_{\mathrm{C}}$ with an approximate level of confidence of 68 %. In order to increase the confidence interval of the estimated parameter, the extended uncertainty, denoted as U, is considered. Then $U = ku_{\mathrm{C}}$, where k stands for the coverage factor, which is generally calculated by estimating the effective degree of freedom in relation with the t-distribution table for a desired confidence level. Typically, k takes the value from 2 to 3. If it is convenient, a relative uncertainty, denoted as u_{r}, is also given and determined as, for example, $u_{\mathrm{r,C}}(y) = u_{\mathrm{C}}(y)/|y|$ for $|y| \neq 0$.

The perception of the measurement accuracy and, at the same time, the measurement uncertainty has changed greatly over the time with the advancement of the experimental techniques and state of knowledge in the data processing. The need to evaluate the vast amount of collected data, especially in the field of astronomy and cartography, spurred the birth of mathematical statistics. It goes back to the middle of 18th century, when Tobias Mayer, a German mathematician, astronomer and selenographer, proposed a method of averages, which he used for the evaluation of his measurements describing the Moon's libration upon observing the Manilius crater and its position changing with time [9–14]. More specifically, the knowledge of the spherical trigonometry help Mayer construct a system of 27 equations with three unknown geometrical parameters, which were to determine. He realized that the accuracy of the estimates could have been increased by replicating the measurements. The first straight idea of solving all possible selected three equations and then considering the averages of all such solutions for three unknown parameters is regarded as very complex due to numerous equation systems. Therefore, Mayer suggested adding equations, leading to a reduction of the problem to three equations with three unknows. For combining the equations to particular groups, he chose an interesting criterion based on maximization of variance of coefficients standing at one unknown parameter (namely, α – see below for details). Besides, he firstly introduced an error of the measurement as an interval, in which the true value of the unknown parameter must lie. Thus, his method provided a determination of estimator accuracy from averages, however, without imposing any criterion. Moreover, his model was greatly simplified with several assumptions and linearization of the original relationship, derived from identities in the spherical geometry. Thus, his estimation of the averages error could have been underestimated or exaugurated, as his initial model showed a non-linearity with non-trivial propagation of the uncertainty.

The scope of the present paper is thus to compare the deviation of the first and second Mayer's procedure for solving a system of equations with the ordinary least-square (OLS) method. The OLS method provides determination of not only the uncertainty of the estimating parameters (similarly as the Mayer's approach) but also the accuracy of the measurements, involving, at the same time, relevant criteria and conditional relationships. In addition, we address the issue of convenience of Mayer's simplification by constructing a non-linear model within the context of modern metrological approach, comparing the results achieved in different variants in the linearized and nonlinearized models. Besides, we estimate the accuracy of the unknown parameters for the selenographic width and length of the Manilius crater and the angle between the true and apparent equator. We also determine the accuracy of Mayer's measurements of the sides of spherical triangles and discuss the results in the context of the uncertainty propagation. The results, presented here, show that despite the conceptual and computational simplification of the Mayer's method, such an approach to data processing can be exploited, with an acceptable level of accuracy, in several practical situations even today.

Description of the computational methodology

Analysis of the task

From the historical perspective, the"rocking" motion of the Moon, known as libration, was firstly studied and described by Tobias Mayer, a German mathematician, astronomer and selenographer, upon evaluation of his vast observations of Moon during the years 1748–1749. He found out that Moon rocks slightly back and forth relative to the line, which connects the centers of the Earth and the Moon. In particular, he focused on finding a relation between the true equator (denoted as QNL in Fig. 1) and true pole (denoted as P in Fig. 1) of the Moon and Moon's apparent equator (denoted as DNB in Fig. 1); the two equators differ as the axis of rotation of Moon is not exactly perpendicular to the orbital plane of the Earth circulating the Sun on its orbit. Here, the apparent Moon's equator is defined as a circle on its surface which is parallel with

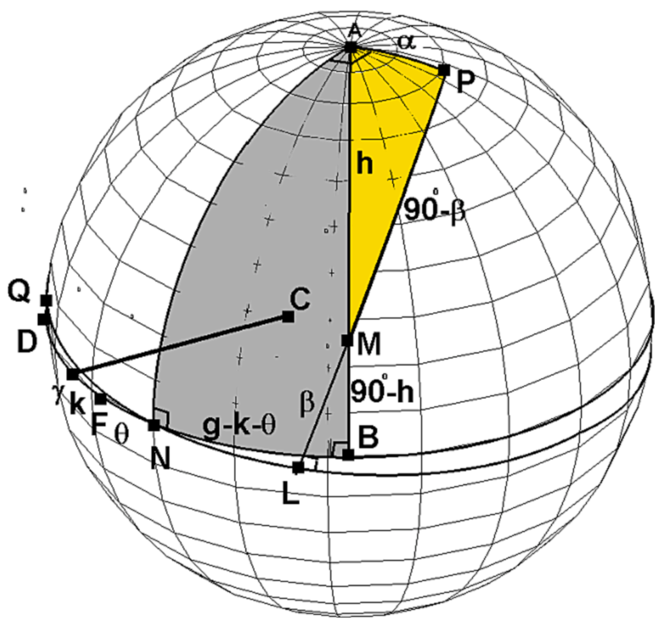

Fig. 1. Schematic representation of the Moon's surface showing important points, apparent and true equators, meridians in apparent and true coordinate systems, and arc lengths used in the Mayer's calculations: M is the position of the crater Manilius, DNB is a part of the apparent Moon's equator with A the apparent Moon's pole and AN and AB the corresponding meridians in the apparent coordinate system, QNL is the part of the true Moon's equator with P the true Moon's pole and PL the corresponding meridian in the true coordinate system, F is the node of the Moon's orbit and the plane of the ecliptic, and γ is the reference point given by the intersection of the equinox from the Moon's center and the apparent equator.

respect to the plane of the Earth's orbit around the Sun; the position of Moon's apparent pole A is determined with respect to the apparent Moon's equator and as viewed by the Earth-based observer oriented by the ecliptic (see Fig. 1). Due to the phenomenon of libration, the circle DNB and the pole A change with time, forming the constantly time-varying coordinate system. On the other hand, the true equator (circle QNL) and true pole (P) are fixed and not observable from the Earth, representing thus the true lunar coordinate system.

In order to derive the relationship between the true and apparent coordinate systems, Mayer chose the crater Manilius (denoted as M in Fig. 1), which he observed on several occasions. Then, lines PL and AB (see Fig. 1) represents meridians of respective polar coordinate systems, both crossing the position of the crater Manilius. In addition, in his calculations, he defined the point γ and F (see Fig. 1), the former taken as the reference point (lying on the circle DNB in the direction of equinox from the center of Moon, C) and the latter defined as the known point of intersection of the plane of the Moon's orbit and apparent equator (circle DNB). At a given time, Mayer measured the arcs $h = $ AM and $g =$ γB. His task was then to the fixed, however unknown, arc length $\alpha =$ AP, the true latitude of the crater Manilius, $\beta =$ ML and the distance θ between the known point F and unknown point N, where N is the point of intersection of the true and apparent equator (see Fig. 1). If we denote k = γF, which is the observed longitude of F, then g, h, and k are observable quantities varying in time due to Moon's libration (and observational error). On the other hand, α, β, and θ are considered as fixed and unknown quantities, which must be determined from the observation. As NAP is a right angle, using the cosine rule and sine rule from spherical trigonometry applied on triangle MAP (yellow spherical triangle in Fig. 1) and BNA (grey spherical triangle in Fig. 1), respectively, one readily arrives at

$$\sin\beta = \cos\alpha\cos h + \sin\alpha\sin h\sin(g - k - \theta) \tag{5}$$

The derived equation is, however, nonlinear with respect to observable (measured) and non-observable (to-be-determined) quantities. In order to find a linear approximation, Mayer adopted several assumptions. From his experience, he knew that α and θ were very small. Thus, cos α, cos θ, and $\cos(\beta - \pi/2 + h)$ were treated as nearly 1 and $\sin(\beta - \pi/2 + h)$ could be written as $\beta - \pi/2 + h$. In addition, he introduced y as $\beta - \pi/2 + h$ as the difference of the arc distances on corresponding meridians between the crater Manilius and apparent and true difference of the arc distances on corresponding meridians between the crater Manilius and apparent and true equator, resulting in $\sin\beta - \cos h = y \sin h$. Then, using angle sum and difference identities, the equation changed to a linear form, i.e.,

$$\beta - 90^\circ + h = \alpha\sin(g - k) - \alpha\sin\theta\cos(g - k) \tag{6}$$

providing approximate relationship between the observable and unknown quantities.

Without a need of linearization, we can straightly construct a regression model, inspired by Mayer's derived Eq. (5). The formula in the form of

$$\sin\beta - x_i\sin\alpha + \sqrt{1 - x_i^2}\sin\alpha\sin\theta - r_i = 0 \tag{7}$$

can be simply rewritten from Equation (A16) in the Appendix A, where $x_i = \sin(g_i - k_i)$, $\sqrt{1 - x_i^2} = \cos(g_i - k_i)$, and $r_i = \beta - 90^\circ + h_i$. For the proof, please, see Appendix A.

While developing his method of lunar distances, Tobias Mayer tackled the problem of solving a system of equations where the number of equations exceeded the number of unknowns. He formulated 27 equations (see Table 1), which he constructed from the observations of the Manilius Crater on the Moon.

At first glance, the characteristics of the Moon's libration motion are not apparent in the system of 27 Mayer's equations. The measured

Table 1
Mayer's set of equations.

Equation number	Equation	Group
1	$\beta - 0.8836\alpha + 0.4682\alpha \sin\theta = 13^\circ10'$	I
2	$\beta - 0.9996\alpha + 0.0282\alpha \sin\theta = 13^\circ8'$	I
3	$\beta - 0.9899\alpha - 0.1421\alpha \sin\theta = 13^\circ12'$	I
4	$\beta - 0.2221\alpha - 0.9750\alpha \sin\theta = 14^\circ15'$	III
5	$\beta - 0.0006\alpha - 1.0000\alpha \sin\theta = 14^\circ42'$	III
6	$\beta - 0.9308\alpha + 0.3654\alpha \sin\theta = 13^\circ1'$	I
7	$\beta - 0.0602\alpha - 0.9982\alpha \sin\theta = 14^\circ31'$	III
8	$\beta + 0.1570\alpha - 0.9876\alpha \sin\theta = 14^\circ57'$	II
9	$\beta - 0.9097\alpha + 0.4152\alpha \sin\theta = 13^\circ5'$	I
10	$\beta - 1.0000\alpha - 0.0055\alpha \sin\theta = 13^\circ2'$	I
11	$\beta - 0.9689\alpha - 0.2476\alpha \sin\theta = 13^\circ12'$	I
12	$\beta - 0.8878\alpha - 0.4602\alpha \sin\theta = 13^\circ11'$	I
13	$\beta - 0.7549\alpha - 0.6558\alpha \sin\theta = 13^\circ34'$	III
14	$\beta - 0.5755\alpha - 0.8178\alpha \sin\theta = 13^\circ53'$	III
15	$\beta - 0.3608\alpha - 0.9326\alpha \sin\theta = 13^\circ58'$	III
16	$\beta - 0.1302\alpha - 0.9915\alpha \sin\theta = 14^\circ14'$	III
17	$\beta + 0.1068\alpha - 0.9943\alpha \sin\theta = 14^\circ56'$	III
18	$\beta + 0.3363\alpha - 0.9418\alpha \sin\theta = 14^\circ47'$	II
19	$\beta + 0.8560\alpha - 0.5170\alpha \sin\theta = 15^\circ56'$	II
20	$\beta - 0.8002\alpha - 0.5997\alpha \sin\theta = 13^\circ29'$	III
21	$\beta + 0.9952\alpha + 0.0982\alpha \sin\theta = 15^\circ55'$	II
22	$\beta + 0.8409\alpha - 0.5412\alpha \sin\theta = 15^\circ39'$	II
23	$\beta + 0.9429\alpha - 0.3330\alpha \sin\theta = 16^\circ9'$	II
24	$\beta + 0.9768\alpha - 0.2141\alpha \sin\theta = 16^\circ22'$	II
25	$\beta + 0.6262\alpha + 0.7797\alpha \sin\theta = 15^\circ38'$	II
26	$\beta + 0.4091\alpha + 0.9125\alpha \sin\theta = 14^\circ54'$	II
27	$\beta - 0.9284\alpha + 0.3716\alpha \sin\theta = 13^\circ7'$	I

values of r and $\times$ (see Eq. (7)) are given by the day of the libration cycle, which is approximately 27.21 days long. In Fig. 2, we have the measurements of r on the left and $\times$ on the right with differentiation of individual groups (i.e., I, II, and III) of Mayer's equations (see Table 1). It can be assumed that the measurements from equations No. 8–18 were realized in the consecutive days. We approximated these data with the function $\theta_1 + \theta_2\sin(2\pi x/27.21 + \theta_3)$. The calculated residual sum of squares allowed us to estimate a standard deviation of 2.96 arc *sec* for the parameter r and a standard deviation of 5.82 arc *sec* for the parameter x. Our estimates of the Mayer's uncertainty measurements are a contribution to the discussion below.

Here, it should be stressed that solving the regression problems can be greatly complicated by multicollinearity. In our case, the correlation coefficient between x and $\sqrt{1-x^2}$ is equal to 0.23. A significantly stronger correlation is found between x and r with a value of 0.99. To solve the regression problems with multicollinearity, we usually use the ridge regression today, for details see the work of Hoerl [18]. Cf. We did not choose this approach because the conditionality numbers of the left-hand matrix in the regression models we used were not very high.

Methods for solving an inconsistent system of equations

The algorithms proposed by Mayer can be viewed as the first attempt of finding the weak solution of a predetermined system of equations. His pioneering works are widely recognized by mathematic historians as impulses launching the so-called first statistical revolution, and his approaches are classified to predecessors of the least-square method. In the numerical study, we show the bias of the Mayer's estimates and point out a huge advantage of the modern approaches, which provide finding not only the estimates of the unknown parameters but also estimating their accuracy, together with the estimation of the accuracy of the measured parameters.

Mayer's system of all possible solvable subgroups of equations

Mayer's first approach was based on the selection of 3 equations out of 27; he suggested to choose those three equations with a criterion of maximizing the differences between the coefficients. However, selecting other three equations gave different estimates of the parameters. As a number of all possible combinations of three equations equals to $\binom{27}{3} = 2925$ at the Mayer's time, it was not feasible to perform all these calculations. Here, we calculated all the 2925 sets of equations and determined the values of the unknown parameters as the average, i.e., center of gravity, of all the solutions for α, β, and θ.

The method is generally applicable to any predetermined set of linear equations. In particular, for a regression line, we would solve all possible pairs of equations, which would be the combination number $\binom{n}{2}$, where n is a number of equations.

Due to errors in the measured parameters, the solutions of these 2925 equations are sometimes quite different, which is due to a combination of errors in some triples of equations. We estimate the uncertainty u_C of the parameters α, β, and θ by taking the square root of the diagonal elements of the sample covariance matrix of the determined solutions.

Hereafter, this computational approach is designated as the Method M1.

Mayer's average method

Mayer's second procedure was based on an idea of dividing 27 equations into three groups, each containing 9 equations (see Table 1). In the first group, he placed nine equations having the largest positive values of the coefficient standing at the unknown α. On the other hand, the second group featured nine equations with the largest negative values of the coefficient at the unknown α. Finally, the last group involved remaining nine equations showing the largest values of the coefficients standing at $\alpha \sin\theta$. In other words, as the coefficient at α and $\alpha \sin\theta$ is $\sin(g-k)$ and $-\cos(g-k)$, respectively, the first set of equations

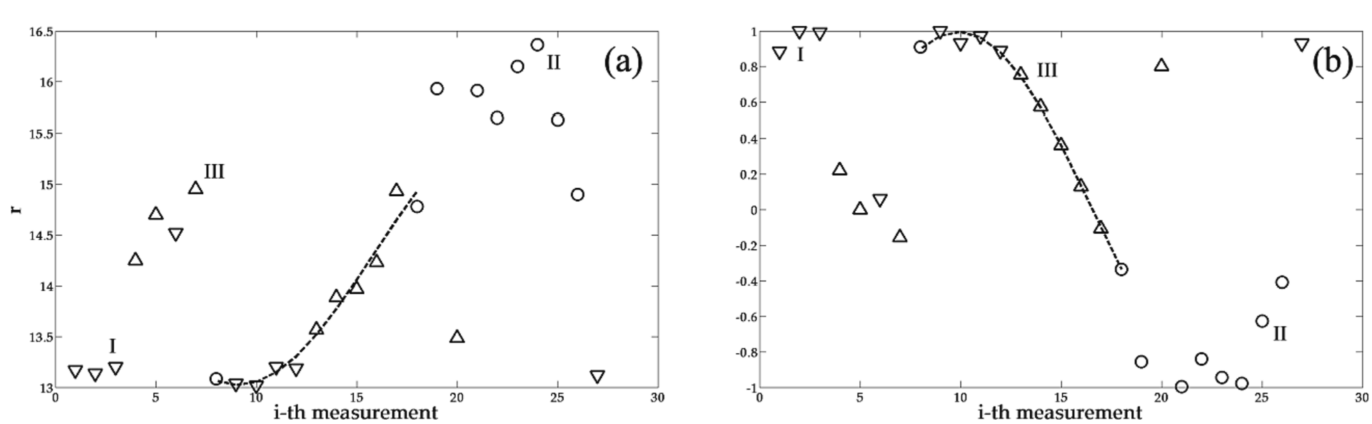

Fig. 2. Values of (a) r and (b) $\times$ from 27 Mayer's equations from consecutive measurements and their fitting. Upside-down triangle, circle, and triangle represent equations from the Group I, II, and III, respectively (see Table 1).

had the coefficients of α very close to 1.0 and the second group of equations showed the coefficients of α very close to – 1.0. In the case of the third group, the values of the coefficients at α oscillated around zero, in addition, with a large value of – $\cos(g-k)$ coefficient at $\alpha\sin\theta$ term. This led to the largest possible differences between the three sums of the respective coefficients, ultimately improving the accuracy of determination of the unknown parameters α, β, and θ.

After grouping proposed by Mayer, we arrive at three equations, which can be written, in the model form, as

$$\sum_{j=1}^{9} 1\cdot\beta - \sum_{j=1}^{9} x_{i_j}\cdot\alpha + \sum_{j=1}^{9}\sqrt{1-x_{i_j}^2}\cdot\alpha\cdot\sin\theta = \sum_{j=1}^{9} r_{i_j} \tag{8}$$

$$\sum_{k=1}^{9} 1\cdot\beta - \sum_{k=1}^{9} x_{i_k}\cdot\alpha + \sum_{k=1}^{9}\sqrt{1-x_{i_k}^2}\cdot\alpha\cdot\sin\theta = \sum_{k=1}^{9} r_{i_k} \tag{9}$$

$$\sum_{l=1}^{9} 1\cdot\beta - \sum_{l=1}^{9} x_{i_l}\cdot\alpha + \sum_{l=1}^{9}\sqrt{1-x_{i_l}^2}\cdot\alpha\cdot\sin\theta = \sum_{l=1}^{9} r_{i_l} \tag{10}$$

where i_j, i_k, i_l run the whole set of indices $i = 1, \ldots, 27$. In accordance with the above-mentioned methodology description, Mayer marked the group i_j with the Roman letter I and placed there the observations for which the coefficient x_{i_j} was the largest positive. He inserted the indices i_k into the Group II, for which the coefficients x_{i_k} were the smallest (negative). Finally, he inserted the remaining indexes i_l into the Group III. Mayer silently assumed that there exist a number of equations that is integer-divisible by the number of unknowns. In his case, $27/3 = 9$.

The method is therefore based on the sum of nine measured values, when the obtained estimate must be divided by the number nine. When reducing the 27 equations to three equations, Mayer calculates the arithmetic mean, which should lead to a reduction in the uncertainty of the estimated parameters. Unfortunately, these are not replicated measurements of the same quantity, as the measurements are carried out on different days of the libration cycle.

Hereafter, this computational model is designated as the Method M2.

Regression model without any simplification

First, for a system of 27 Mayer equations, let us build a theoretical model using relation (A16), i.e.,

$$r_i = f(x_i,\theta) = \beta - x_i\sin\alpha + \sqrt{1-x_i^2}\sin\alpha\sin\theta \tag{11}$$

where $\theta = (\alpha,\ \beta,\theta)^{'}, \mathbf{r} = f(r_1,\cdots,r_n)^{'} = (\pi/2-h_1,\cdots,\pi/2-h_n)^{'}$, and $\mathbf{x} = (x_1,\cdots,x_n)' = (\sin(g_1-\ k_1),\ \cdots,\ \sin(g_n-\ k_n))'$. In this model, we directly calculate the estimates of α, β, and θ.

Suppose that the vector $\mathbf{x}$ is given deterministically. Under this assumption, we can move from the theoretical model to the stochastic measurement model, $\mathbf{Y}\sim \mathrm{N}(\mathbf{f}(\mathbf{x},\theta),\Sigma_\mathrm{B})$, where $\mathbf{Y} = (Y_1,\cdots,Y_n)'$, $\Sigma_\mathrm{B} = \sigma^2\,\mathbf{I}$, i.e., that $\mathbf{Y} = \mathbf{f}(\mathbf{x},\theta)+\boldsymbol{\varepsilon} = \mathbf{r}+\boldsymbol{\varepsilon}$, where the measurement error $\boldsymbol{\varepsilon}$ has zero mean $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\mathrm{var}(\boldsymbol{\varepsilon}) = \Sigma_\mathrm{B}$. Thus, in our model, we assume that there is no intercorrelation of the measurement errors between quantities Y_i, $i = 1,\ldots, n$.

We can linearize the model as follows

$$\mathbf{Y}\sim\mathbf{Y}_0+\mathbf{F}\delta\theta, \tag{12}$$

$$\mathbf{F} = \frac{\partial\mathbf{f}(\mathbf{x},\theta_0)}{\partial\theta}, \delta\theta = \theta-\theta_0, \tag{13}$$

where θ_0 is the appropriate initial solution. Then

$$\boldsymbol{Y}-\boldsymbol{Y}_0 = \boldsymbol{F}(\boldsymbol{\theta}-\boldsymbol{\theta}_0) = \boldsymbol{F}\delta\boldsymbol{\theta}. \tag{14}$$

We calculate the $\widehat{\theta}$ estimate using the least squares method with the relation in the form of

$$\delta\widehat{\boldsymbol{\theta}} = \left(\boldsymbol{F}^{'}\boldsymbol{\Sigma}_\mathbf{B}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^{'}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{Y}-\boldsymbol{Y}_0),\ \widehat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0+\delta\widehat{\boldsymbol{\theta}}\ . \tag{15}$$

Its accuracy is given by the formula

$$\mathrm{var}_\mathrm{C}(\widehat{\boldsymbol{\theta}}) = \left(\boldsymbol{F}^{'}\boldsymbol{\Sigma}_\mathrm{B}^{-1}\boldsymbol{F}\right)^{-1}. \tag{16}$$

In our case,

$$\mathbf{F} = \begin{pmatrix} -x_1\cos\alpha+\sqrt{1-x_1^2}\cos\alpha\sin\theta & 1 & \sqrt{1-x_1^2}\sin\alpha\cos\theta \\ \vdots & \vdots & \vdots \\ -x_n\cos\alpha+\sqrt{1-x_n^2}\cos\alpha\sin\theta & 1 & \sqrt{1-x_n^2}\sin\alpha\cos\theta \end{pmatrix}. \tag{17}$$

After calculating $\widehat{\theta}$, we can determine $\widehat{\mathbf{Y}} = f(\mathbf{x},\widehat{\theta})$ and residual sum of squares $S_\mathrm{res} = \sum_{i=1}^{n}(y_i-\widehat{y}_i)^2 = \sum_{i=1}^{n}e_i^2$. The estimate of measurement accuracy $\mathbf{Y}$ is then given by $s^2 = \widehat{\sigma}^2 = S_\mathrm{res}/(n-3)$.

The estimate $\gamma = \alpha\sin\theta$ and its variance is calculated in the Appendix B by use of the Taylor expansion of the first and second order.

Hereafter, this model is designated as the Method M3.

Regression model with Mayer's simplification

Now, for a system of 27 Mayer equations, it is possible to use Mayer simplification (see Table 1), i.e.,

$$r_i = f(x_i,\theta) = \beta - x_i\alpha + \sqrt{1-x_i^2}\gamma \tag{18}$$

with $\gamma = \alpha\sin\theta$, where $\theta = (\alpha,\ \beta,\gamma)^{'}$, $\boldsymbol{r} = f(r_1,\cdots,r_n)^{'} = (\pi/2-h_1,\cdots,\pi/2-h_n)^{'}$, and $\mathbf{x} = (x_1,\cdots,x_n)' = (\sin(g_1-\ k_1),\ \cdots,\ \sin(g_n-\ k_n))'$. In this model, we thus directly calculate the estimates α, β, and γ. However, the estimate of θ is not at our disposal. Therefore, we must estimate $\boldsymbol{\theta}$ indirectly using the formula of $\gamma = \alpha\sin\theta$, i.e., $\theta = \arcsin(\gamma/\alpha)$. To do so, we need to apply the Taylor expansion for the calculation of the expected value for θ and its standard deviation. For more details, please see Appendix C.

Adopting the above-discussed Mayer's approach, we get the matrix in the form of

$$\mathbf{F} = \begin{pmatrix} -x_1 & 1 & \sqrt{1-x_1^2} \\ \vdots & \vdots & \vdots \\ -x_n & 1 & \sqrt{1-x_n^2} \end{pmatrix} \tag{19}$$

In this model, we use OLS estimate in the form of

$$\widehat{\boldsymbol{\theta}} = \left(\boldsymbol{F}^{'}\boldsymbol{\Sigma}_\mathbf{B}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^{'}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{Y}\ . \tag{20}$$

Estimate of the covariance matrix is again given by the formula (16).

Hereafter, this model is designated as the Method M4.

Results and discussion

Mayer's approaches

To solve the equations, Mayer initially used the method of selected points. He chose 3 equations out of 27 in such a way, so as to maximize differences among coefficients. This was supposed to ensure good results on the unknowns. By choosing the equations No. 9, 16 and 19 from Table 1, he obtained the following set of equations:

$\beta + 0.9097\alpha + 0.4152\alpha\sin\theta = 13^\circ5'$ (No. 9 from eq. system I).

$\beta + 0.1302\alpha + 0.9915\alpha\sin\theta = 14^\circ14'$ (No. 16 from eq. system II).

$\beta - 0.8560\alpha + 0.5170\alpha\sin\theta = 15^\circ56'$ (No. 19 from eq. system III).

Solving this set of equations yields $\theta_1 = \alpha = 1^\circ38'$, $\theta_2 = \beta = 14^\circ32'$, and $\theta_3 = \alpha\sin\theta = 0^\circ6'$. Then, $\theta = \arcsin(\theta_3/\theta_1) = 0^\circ4'$. This solution is graphically depicted by a symbol of cyan pentagram in Fig. 3.

However, Mayer was aware that the measurement error and random choice significantly influence the resulting estimates. He thus came to a conclusion that his method was unsatisfactory, as choosing another three equations would give a different estimate. An ideal way to proceed with this task would be to consider all possible combinations of three

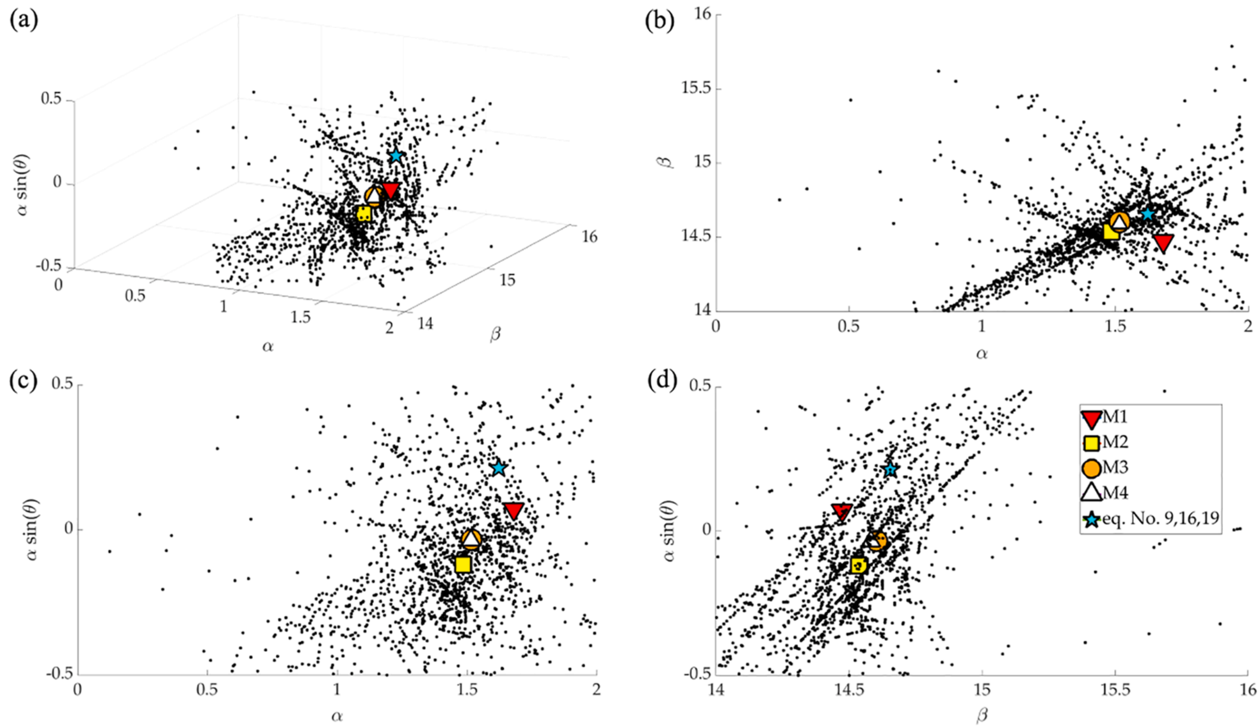

Fig. 3. Calculated estimates (a) in the 3D space of parameters α, β, and $\alpha \sin \theta$, (b) in the plane of parameters α and β, (c) in the plane of parameters α and $\alpha \sin \theta$, and (d) in the plane of parameters β and $\alpha \sin \theta$.

equations out of 27 in total and then average the results. However, this would involve $\binom{27}{3} = 2925$ sets of equations. No wonder that Mayer gave up, rejecting the method as too laborious.

From 2925 solutions, it was not possible to determine the angle θ for 356 systems of equations as the solutions for θ_1 and θ_3 led to abs(θ_3/θ_1) > 1. Then, arcsin(θ_3/θ_1) does not exist in the real value domain. For example, this happened for the set of three equations numbered 1, 4, and 16. The values in parentheses are calculated from the permissible 2569 solutions (see Table 2). Thus, from our calculation, the average of all the permissible 2569 solutions (by Method A) is $\theta_1 = \alpha = 1°34'$, $\theta_2 = \beta = 14°29'$, and $\theta_3 = \alpha \sin \theta = 0°4'$ (see Table 2). Then, $\theta = \arcsin(\theta_3/\theta_1) =$ -2°34′. This solution is graphically depicted by symbol of red triangle in Fig. 3.

Instead, he divided 27 equations into three groups of 9 equations each (see Table 1; Method M2). In each group, he summed the equations and obtained the following three equations, i.e.,

$9\beta - 8.4987\alpha + 0.7932\alpha \sin \theta = 118°8'$ (sum of eq. system I),

$9\beta + 6.1404\alpha - 1.7443\alpha \sin \theta = 140°17'$ (sum of eq. system II),

$9\beta - 2.7977\alpha - 7.9149\alpha \sin \theta = 127°32'$ (sum of eq. system III).

When solving, we get $\theta_1 = \alpha = 1°29'$, $\theta_2 = \beta = 14°32'$, and $\theta_3 = \alpha \sin \theta = -0°7'$ (see Table 2). Then, $\theta = \arcsin(\theta_3/\alpha) = -4°37'$. These estimates, calculated by the Mayer method of averages, are graphically shown by a symbol of yellow square in Fig. 3.

Estimates from the regression models

In Section 2, two regression models, labelled as M3 and M4, were constructed to solve the unknows $\theta_1 = \alpha$, $\theta_2 = \beta$, and $\theta_3 = \alpha \sin \theta$ (with $\theta = \arcsin(\theta_3/\alpha)$). Within these models, we consider a covariance matrix of the vector **Y** in the form of $\sigma^2\mathbf{I}$, where $\sigma = 2' = 0.0005818$ [rad]. Our choice for the σ-value can be justified based on the two facts, i.e. (i) historical literature and (ii) estimate of the standard deviation of the measurement of the parameter x and r based on data approximation by a periodic function. From the correspondence letters between Tobias Mayer and Leonhard Euler [15], it can be deduced that the accuracy of the Mayer's lunar tables (published after the year 1750), used as a reliable tool for navigation in the sea voyages during the 2nd half of the 17th century, was claimed to be around 30 arc *sec*. Here, the error associated with the Mayer's measurement of Moon's libration is not explicitly mentioned (done before 1750). Moreover, in the historical study on the measurement precision by Blitzstein [16], it can be found that John Flamsteed, a famous English astronomer of the late 16th and early 17th century, was able to estimate star positions with an accuracy lying in the interval from 9 to 50 arc *sec*. These measurement uncertainties are in good agreement with the other reference from the work by Wallis [17], who mentions the accuracy of 10 arc *sec* in John Flamsteed star measurements in 17th century, i.e., the same time period when Tobias Mayer performed his research on the Lunar libration. Moreover, from our periodicity analysis of the measured parameters x_i and r_i ($i = 8, \ldots, 18$) in the Mayer's equations (see Fig. 2), we estimated a standard deviation of 2.96 arc *sec* for the parameter r and a standard deviation of 5.82 arc *sec* for the parameter x. Thus, considering these findings together with the physical/technical aspects of the Moon's observation including the light refraction, telescope/instrumental errors, etc., the measurement error of 2′ seems quite reasonable.

The results from the Method M1, M2, M3, and M4 are summarized in Table 2. In Fig. 3, the solutions within each method are graphically depicted by a symbol of red triangle, yellow square, orange circle, and white triangle for the Method M1, M2, M3, and M4, respectively. Please, see the Appendix B and C for more details on derivation of the mean value and error expressions with the Taylor series for the parameters $\gamma_{(1)}$, $\gamma_{(2)}$ and $\theta_{(1)}$, $\theta_{(2)}$, used in the Method M3 and M4, respectively (the index implies the order of the Taylor expansion (1 and 2)).

Calculation of estimates and uncertainties in Tab. 2 was based on the estimates of the covariance matrices obtained by the OLS method. Specifically, the following matrices were determined:

$$\mathrm{M1}: \widehat{\mathrm{cov}}(\alpha, \beta, \gamma) = \begin{pmatrix} 0.0293 & 0.0294 & -0.0112 \\ 0.0294 & 0.0338 & -0.0114 \\ -0.0112 & -0.0114 & 0.0051 \end{pmatrix} [\mathrm{rad}^2], \quad (21)$$

Table 2
Values of the estimates of the unknown parameters α, β, α sin θ, and θ, computed within the Method M1, M2, M3, and M4, where Type stands for the uncertainty type, u_C is the combined uncertainty, U represents the extended uncertainty, and $u_{r,C}$ denotes the relative uncertainty. Note: Calculations were done for a diagonal matric in the form of $\Sigma_B = \left(\frac{2}{60}\right)^2 \mathbf{I}$, i.e., for $u_B = 2'$.

Method	Parameter	Estimate	Type	u_C	$U = 1.96\ u_C$	$u_{r,C}$
M1♥	α	1°34′19″ (1°40′45″)	C = A + B	9°14′ (9°49′)	18°06′ (19°14′)	5,87 (5,84)
M1♥	β	14°28′ (14°28′24″)	C = A + B	10°8′ (10°32′)	19°52′ (20°39′)	0,70 (0,73)
M1♥	γ	0°7′45″ (0°4′15″)	C = A + B	4°32′ (4°05′)	8°52′ (8°01′)	35,04 (57,47)
M1♥	θ	NaN (-2°33′47″)†	C♣	NaN (19°9′)♣	NaN (37°32′)♣	NaN (7,47) ♣
M2	α	1°29′3″	B	NaN♠	NaN♠	NaN♠
M2	β	14°32′18″	B	NaN♠	NaN♠	NaN♠
M2	γ	−0°7′10″	B	NaN♠	NaN♠	NaN♠
M2	θ	−4°37′18″	B	NaN♠	NaN♠	NaN♠
M3	α	1°31′0″	C = A + B	0°0′33″	0°1′05″	0.006
M3	β	14°36′18″	C = A + B	0°0′49″	0°1′36″	0.001
M3	θ	−1°17′54″	C = A + B	0°40′54″	1°20′10″	0.525
M3	$\gamma_{(1)}$	−0°2′4″ †	C	0°1′9″◆	0°2′16″	0.564
M3	$\gamma_{(2)}$	−0°2′4″ ††	C	0°1′9′◆◆	0°2′16″	0.564
M4	α	1°30′58″	C = A + B	0°0′33″	0°1′05″	0.006
M4	β	14°36′18″	C = A + B	0°0′49″	0°1′36″	0.001
M4	γ	−0°2′9″	C = A + B	0°1′11″	0°2′19″	0.550
M4	$\theta_{(1)}$	−1°38′17″*	C	0°1′27″♣	0°2′52″	0.012
M4	$\theta_{(2)}$	−1°38′17″ **	C	0°1′27″♣♣	0°2′53″	0.012

♥NaN denotes undefined value, average denotes the sample average, and *s* represents the sample standard deviation. In the Method M1, the values without brackets are calculated from 2925 systems of equations. In the Method M1, values in the brackets are calculated from permissible 2569 systems of equations as the argument of the arcsin function, i.e., $\theta = \arcsin(\gamma/\alpha)$, must be in the interval from – 1 to 1. However, this was not fulfilled for 44 systems of equations.

♣Following Eq. (3), we get $\hat{\sigma}_\theta^2 = \left(-\frac{\hat{\gamma}}{1-\sqrt{\frac{\hat{\alpha}^2}{\hat{\gamma}^2}}}\right)^2 \hat{\sigma}_\alpha^2 + \left(-\frac{1}{1-\sqrt{\frac{\hat{\alpha}^2}{\hat{\gamma}^2}}}\right)^2 \hat{\sigma}_\gamma^2 + 2\left(\frac{\hat{\gamma}}{\left(1-\sqrt{\frac{\hat{\alpha}^2}{\hat{\gamma}^2}}\right)^2}\right)\hat{\sigma}_{\alpha\gamma}$.

♠The estimate in the Method M2 was determined unambiguously. The solution of the system of 3 equations with 3 unknowns does not provide space for the calculation of A-type uncertainty. The option was to estimate the uncertainty using the Monte Carlo simulation method.

◆The standard deviation of $\gamma = \alpha \sin \theta$ is calculated following the Eq. (3), i.e., as $\sigma_\gamma^2 = (\sin\hat{\theta})^2\sigma_\alpha^2 + (\hat{\alpha}\cos\hat{\theta})^2\sigma_\theta^2 + 2(\sin\hat{\theta}\hat{\alpha}\cos\hat{\theta})\sigma_{\alpha\theta}$, where $\hat{\alpha}$ and $\hat{\theta}$ are mean values of α and θ and $\hat{\sigma}_\alpha$ and $\hat{\sigma}_\theta$ stand for their standard deviation estimates, respectively. Cf. (C6).

†Calculated from Eq. (C2). $\theta_{(1)}$ and $\theta_{(2)}$ are estimates θ from the Taylor series expansion. For more details, please see Appendix C.

†† Calculated from Eq. (C4).

◆ Calculated from Eq. (C3).

◆◆ Calculated from Eq. (C5).

* Calculated from Eq. (B2).

** Calculated from Eq. (B4).

♣ Calculated from Eq. (B3).

♣♣ Calculated from Eq. (B5).

*Values listed in the last column were calculated by the simple mathematical procedure, described in Introduction (please, see text following formula (3)).

$$\text{M3}: \widehat{\text{cov}}(\alpha,\ \beta, \theta) = \begin{pmatrix} 0.0003 & 0.0001 & -0.0044 \\ 0.0001 & 0.0006 & -0.0249 \\ -0.0044 & -0.0249 & 1.4159 \end{pmatrix} 10^{-4}\left[\text{rad}^2\right], \quad (22)$$

$$\text{M4}: \widehat{\text{cov}}(\alpha,\ \beta, \gamma) = \begin{pmatrix} 0.0259 & 0.0125 & -0.0129 \\ 0.0125 & 0.0574 & -0.0725 \\ -0.0129 & -0.0725 & 0.1197 \end{pmatrix} 10^{-6}\left[\text{rad}^2\right] \quad (23)$$

Discussion

From the results, it is obvious that the bias of Mayer's first approach (i.e., Method M1) is considerable. The calculated standard deviations of the estimates are huge; this can be understood in terms of a combination of errors in some systems of triple equations. The second method (i.e., Method M2), the method of averages, provided an estimate very close to the estimates of α, β, and α sin θ, given by the Method M3 and M4. However, the error propagation in $\theta = \arcsin(\theta_3/\alpha)$ leads to a large bias of the θ estimate in the Method M1.

The Methods M3 and M4 provide very similar estimates of all the parameters. It is seen that in the Method M4, the replacement of the value of sin α by α and cos α by 1 did not resulted in any significant deviation of the estimated parameters in comparison to those obtained from the Method M3.

In the Method M4, the problem of calculation of θ is difficult, especially due to the non-linear relationship appearing in the formulation of this model. Estimates based on the Taylor's expansion had to be used here. As seen from the results in Table 2, it turned out that the solutions involving either the 1st or 2nd Taylor expansion order were almost identical.

The OLS estimates in the Methods M3 and M4 show comparable relative errors for the α and θ parameters.

Conclusion

In this study, we presented a thorough analysis of various mathematical strategies applied for determining the unknown parameters and their accuracies in the problem of lunar measurements, performed by Tobias Mayer in the middle of 18th century. We proved that if the relationships among measured and unknown quantities are not simplified, the estimates of the unknown parameters amounts to values with the smallest standard deviations, implying that their true value lies in a much narrow interval compared to other approaches which used a various level of simplifying assumptions. Nevertheless, if we compare the estimates of the unknown parameters within various studied models, the values do not vary dramatically. The only unknown quantity that shows an appreciable deviance within the models, is $\alpha\sin\theta$, most due to a non-linear behavior between α and θ.

Apart from estimating unknown parameters, both Mayer's approaches, discussed within this study, do not provide an estimate of their accuracy. His method of averages was replaced by the OLS method about 50 years later, which is based on minimizing the residual sum of squares. In addition, the OLS method allows determining the accuracy of estimates of both unknown parameters and measurements. The Mayer's approach does not adopt any criterion for solution finding as other predecessors of the OLS method such as the Boscovich method and the Laplace method. The reason why the OLS method had gained a privileged position in the field of searching for the estimation of the regression line lies in a simple generalization for more complex models and possibility of estimating the variation matrices of the calculated parameters and the standard deviation of the measurement.

In summary, it should be highlighted that the Mayer's method of averages became the first statistical method for solving the regression problem, although this solution was not based on the minimization of a suitable criterion as mentioned above. The Mayer's innovative solution to reduce the influence of metrological uncertainties on the estimated parameters should be thus appreciated and can be thus used in modern practical approaches without a significant loss of reliability of the results. Mayer's method can be thus very powerful and practically suitable, especially in cases when we do not require the knowledge of the accuracy of the estimated parameters.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgements

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Appendix A. Derivation of the simplified Mayer's model

If we apply a cosine identity in a spherical trigonometry form on a triangle MAP (see Fig. 4), we get

$$\cos\left(\frac{\pi}{2}-\beta\right) = \cos\alpha\cos h + \sin\alpha\sin h\cos a \quad \text{(A1)}$$

i.e.,

$$\sin\beta = \cos\alpha\cos h + \sin\alpha\sin h\cos a \quad \text{(A2)}$$

However, as a triangle NAP is a right-angle triangle, then $a = \frac{\pi}{2} - x$. Applying a sine rule on a triangle NAB, we arrive at

$$\frac{\sin x}{\sin(g-k-\theta)} = \frac{\sin\frac{\pi}{2}}{\sin\left(h+\frac{\pi}{2}-h\right)} \quad \text{(A3)}$$

hence

$$\sin x = \sin(g-k-\theta) \quad \text{(A4)}$$

Then

$$\sin\beta = \cos\alpha\cos h + \sin\alpha\sin h\cos\left(\frac{\pi}{2}-x\right) \quad \text{(A5)}$$

$$\sin\beta = \cos\alpha\cos h + \sin\alpha\sin h\sin x \quad \text{(A6)}$$

$$\sin\beta = \cos\alpha\cos h + \sin\alpha\sin h\sin(g-k-\theta) \quad \text{(A7)}$$

In order to linearize the equation (A7), we firstly express $\sin(g-k-\theta)$ in a simpler form. As $\sin(g-k-\theta) = \sin(g-k)\cos\theta - \sin\theta\cos(g-k)$ and α and θ are very small, implying that $\sin\alpha \approx \alpha$, $\sin\theta \approx \theta$, $\cos\alpha \approx 1$, and $\cos\theta \approx 1$, then $\sin(g-k-\theta) \approx \sin(g-k) - \sin\theta\cos(g-k)$. The equation (A8) can be then rewritten as

$$\sin\beta = \cos h + \sin\alpha\sin h[\sin(g-k) - \sin\theta\cos(g-k)] \quad \text{(A8)}$$

or alternatively

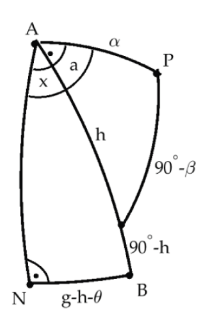

Fig. 4. Geometric scheme used by Mayer for simplification of his model.

$$\sin\beta - \cos h = \sin\alpha\sin h\sin(g-k) - \sin\alpha\sinh\sin\theta\cos(g-k) \tag{A9}$$

Let us introduce a variable $y = \beta - \left(\frac{\pi}{2} - h\right)$, which denotes a difference of distances of the crater Manilius from true and apparent equator (see Fig. 1). Considering that y is a very small value, then

$$\sin\beta = \sin\left(y - h + \frac{\pi}{2}\right) = \sin(y-h)\cos\frac{\pi}{2} + \cos(y-h)\sin\frac{\pi}{2} = \cos(y-h) = \cos(h-y), \tag{A10}$$

$$\sin\beta = \cos h\cos y + \sin h\sin y, \tag{A11}$$

$$\sin\beta \approx \cos h + y\sin h, \tag{A12}$$

as $\sin y \approx y$ and $\cos y \approx 1$. Then

$$\sin\beta - \cos h = y\sin h. \tag{A13}$$

Using the relationship (A13) in equation (A7), we get.
$y\sin h = \sin\alpha\sin h\sin(g-k) - \sin\alpha\sin h\sin\theta\cos(g-k)$, (A14)

$$y = \sin\alpha\sin(g-k) - \sin\alpha\sin\theta\cos(g-k), \tag{A15}$$

$$\beta - \frac{\pi}{2} + h = \alpha\sin(g-k) - \alpha\sin\theta\cos(g-k), \tag{A16}$$

which is the linearized model of the Mayer's measurements.

Appendix B. Derivation of the error expressions in the method M3

Let us consider quantities defined by α, β, and θ. Let $\gamma = \alpha\sin(\theta)$. The standard deviations $\sigma_\alpha, \sigma_\beta, \sigma_\theta$ and the values of $\mathrm{E}(\alpha) = \alpha_0, \mathrm{E}(\beta) = \beta_0, \mathrm{E}(\theta) = \theta_0$ are known. Our task is to find $\mathrm{E}(\gamma)$ and $\mathrm{var}(\gamma)$. To determine them, partial derivatives are necessary.

$$\frac{\partial\gamma}{\partial\alpha} = \sin(\theta),\ \frac{\partial\gamma}{\partial\theta} = \alpha\cos(\theta),\ \frac{\partial^2\gamma}{\partial\alpha^2} = 0,\ \frac{\partial^2\gamma}{\partial\theta^2} = -\alpha\cos(\theta), \frac{\partial^2\gamma}{\partial\alpha\partial\theta} = \cos(\theta). \tag{B1}$$

Firstly, we use the Taylor series in the first order, i.e.,

$$\mathrm{E}\left(\gamma_{(1)}\right) = \alpha_0\sin\left(\theta_0\right), \tag{B2}$$

$$\mathrm{var}\left(\gamma_{(1)}\right) = \left(\frac{\partial\gamma}{\partial\alpha}\right)^2(\alpha_0,\beta_0,\theta_0)\sigma_\alpha^2 + \left(\frac{\partial\gamma}{\partial\theta}\right)^2(\alpha_0,\beta_0,\theta_0)\sigma_\theta^2 + \left(\frac{\partial\gamma}{\partial\alpha}\right)\left(\frac{\partial\gamma}{\partial\theta}\right)(\alpha_0,\beta_0,\theta_0)\sigma_{\alpha\theta}. \tag{B3}$$

If we use the Taylor series in the second order, we can then correct the values of the average and variance

$$E\left(\gamma_{(2)}\right) = \alpha_0 sin\theta_0 + \frac{1}{2}\left[\frac{\partial^2\gamma}{\partial\alpha^2}(\alpha_0,\beta_0,\theta_0)\sigma_\alpha^2 + 2\frac{\partial^2\gamma}{\partial\alpha\partial\theta}(\alpha_0,\beta_0,\theta_0)\sigma_{\alpha\theta} + \frac{\partial^2\gamma}{\partial\theta^2}(\alpha_0,\beta_0,\theta_0)\sigma_\theta^2\right]. \tag{B4}$$

$$\begin{aligned}\mathrm{var}\left(\gamma_{(2)}\right) = {}& \left(\frac{\partial\gamma}{\partial\alpha}(\alpha_0,\beta_0,\theta_0)\right)^2\sigma_\alpha^2 + \left(\frac{\partial\gamma}{\partial\theta}(\alpha_0,\beta_0,\theta_0)\right)^2\sigma_\theta^2 + \frac{1}{4}\left[\left(\frac{\partial^2\gamma}{\partial\alpha^2}(\alpha_0,\beta_0,\theta_0)\right)^2(\mathrm{M}_4\alpha - \sigma_\alpha^4) + \left(\frac{\partial^2\gamma}{\partial\theta^2}(\alpha_0,\beta_0,\theta_0)\right)^2(\mathrm{M}_4\theta - \sigma_\theta^4) + \left(\frac{\partial^2\gamma}{\partial\alpha\partial\theta}(\alpha_0,\beta_0,\theta_0)\right)^2\sigma_\alpha^2\sigma_\theta^2\right.\\ &\left. + \frac{\partial\gamma}{\partial\alpha}(\alpha_0,\beta_0,\theta_0)\frac{\partial^2\theta}{\partial\alpha^2}(\alpha_0,\beta_0,\theta_0)\mathrm{M}_3\alpha + \frac{\partial\gamma}{\partial\theta}(\alpha_0,\beta_0,\theta_0)\frac{\partial^2\gamma}{\partial\theta^2}(\alpha_0,\beta_0,\theta_0)\mathrm{M}_3\theta\right].\end{aligned} \tag{B5}$$

Cf. [17, p. 07]. If we assume the normality of the measurement errors for α and γ, the moments of the third order are equal to zero, i.e., $\mathrm{M}_3\alpha = \mathrm{M}_3\theta = 0$ and $\mathrm{M}_4\alpha = 3\sigma_\alpha^4, \mathrm{M}_4\theta = 3\sigma_\theta^4$.

Appendix C. Derivation of the error expressions in the method M4

Let us consider quantities defined by $\theta_1 = \alpha$, $\theta_2 = \beta$, and $\theta_3 = \alpha\sin\theta = \gamma$. Then $\theta = \arcsin(\gamma/\alpha)$. The standard deviations $\sigma_\alpha, \sigma_\beta, \sigma_\gamma$ and the values of $\mathrm{E}(\alpha) = \alpha_0, \mathrm{E}(\beta) = \beta_0, \mathrm{E}(\gamma) = \gamma_0$ are known. Our task is to find $\mathrm{E}(\theta)$ and $\mathrm{var}(\theta)$. , We simply calculate

$$\begin{aligned}\frac{\partial\theta}{\partial\alpha} &= \frac{1}{\sqrt{1-\frac{\gamma^2}{\alpha^2}}}\left(-\frac{\gamma}{\alpha^2}\right) = -\frac{\gamma}{\alpha\sqrt{\alpha^2-\gamma^2}}, \frac{\partial\theta}{\partial\gamma} = \frac{1}{\sqrt{1-\frac{\gamma^2}{\alpha^2}}}\frac{1}{\alpha} = \frac{1}{\sqrt{\alpha^2-\gamma^2}}, \frac{\partial^2\theta}{\partial\alpha^2} = -\gamma\left[-\alpha^{-2}(\alpha^2-\gamma^2)^{-\frac{1}{2}} + \alpha^{-1}\left(-\frac{1}{2}\right)(\alpha^2-\gamma^2)^{-\frac{3}{2}}2\alpha\right]\\ &= \frac{\gamma}{\alpha^2\sqrt{\alpha^2-\gamma^2}} - \frac{\gamma}{(\alpha^2-\gamma^2)\sqrt{\alpha^2-\gamma^2}}, \frac{\partial^2\theta}{\partial\gamma^2} = -\frac{1}{2}(\alpha^2-\gamma^2)^{-\frac{3}{2}}(-2\gamma) = \frac{\gamma}{(\alpha^2-\gamma^2)\sqrt{\alpha^2-\gamma^2}}, \frac{\partial^2\theta}{\partial\alpha\partial\gamma} = -\frac{1}{2}(\alpha^2-\gamma^2)^{-\frac{3}{2}}(2\alpha) = -\frac{\alpha}{(\alpha^2-\gamma^2)\sqrt{\alpha^2-\gamma^2}}\end{aligned} \tag{C1}$$

Now, we use the Taylor series in the first order, i.e.,

$$\mathrm{E}\left(\theta_{(1)}\right) = arcsin\frac{\gamma_0}{\alpha_0}, \tag{C2}$$

$$\mathrm{var}\left(\theta_{(1)}\right) = \left(\frac{\partial\theta}{\partial\alpha}\right)^2(\alpha_0,\beta_0,\gamma_0)\sigma_\alpha^2 + \left(\frac{\partial\theta}{\partial\gamma}\right)^2(\alpha_0,\beta_0,\gamma_0)\sigma_\gamma^2 \tag{C3}$$

Further

$$\mathrm{E}\left(\theta_{(2)}\right) = arcsin\frac{\mathrm{E}(\gamma)}{\mathrm{E}(\alpha)} + \frac{1}{2}\left[\frac{\partial^2\theta}{\partial\alpha^2}(\alpha_0,\beta_0,\gamma_0)\sigma_\alpha^2 + 2\frac{\partial^2\theta}{\partial\alpha\partial\gamma}(\alpha_0,\beta_0,\gamma_0)\sigma_{\alpha\gamma} + \frac{\partial^2\theta}{\partial\gamma^2}(\alpha_0,\beta_0,\gamma_0)\sigma_\gamma^2\right], \tag{C4}$$

$$\mathrm{var}\left(\theta_{(2)}\right) = \left(\frac{\partial\theta}{\partial\alpha}(\alpha_0,\beta_0,\gamma_0)\right)^2\sigma_\alpha^2 + \left(\frac{\partial\theta}{\partial\gamma}(\alpha_0,\beta_0,\gamma_0)\right)^2\sigma_\gamma^2 + \frac{1}{4}\left[\left(\frac{\partial^2\theta}{\partial\alpha^2}(\alpha_0,\beta_0,\gamma_0)\right)^2\left(\mathrm{M}_4\alpha - \sigma_\alpha^4\right) + \left(\frac{\partial^2\theta}{\partial\gamma^2}(\alpha_0,\beta_0,\gamma_0)\right)^2\left(\mathrm{M}_4\gamma - \sigma_\gamma^4\right) + \left(\frac{\partial^2\theta}{\partial\alpha\partial\gamma}(\alpha_0,\beta_0,\gamma_0)\right)^2\sigma_\alpha^2\sigma_\gamma^2 + \frac{\partial\theta}{\partial\alpha}(\alpha_0,\beta_0,\gamma_0)\frac{\partial^2\theta}{\partial\alpha^2}(\alpha_0,\beta_0,\gamma_0)\mathrm{M}_3\alpha + \frac{\partial\theta}{\partial\gamma}(\alpha_0,\beta_0,\gamma_0)\frac{\partial^2\theta}{\partial\gamma^2}(\alpha_0,\beta_0,\gamma_0)\mathrm{M}_3\gamma\right]. \tag{C5}$$

For $\alpha_0 = 1^\circ 31'$, $\beta_0 = 14^\circ 36'$, and $\gamma_0 = -2' 9''$, we then get

Cf. [17, p. 607]. If we assume the normality of the measurement errors for α and γ, the moments of the third order are equal to zero, i.e., $\mathrm{M}_3\alpha = \mathrm{M}_3\gamma = 0$ and $\mathrm{M}_4\alpha = 3\sigma_\alpha^4, \mathrm{M}_4\gamma = 3\sigma_\gamma^4$.

References

[1] Bureau International des poiuds et Mesures. International Vocabulary of Metrology—Basic and General Concepts and Associated Terms (VIM). Paris: Joint Committee for Guides in Metrology; 2012.

[2] Conroy RS. Frequency standards, metrology and fundamental constants. Contemp Phys 2003;44(2):99–135.

[3] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML 2008 Evaluation of Measurement Data – Guide to the Expression of Uncertaint in Measurement GUM 1995 with minor corrections. Joint Committee for Guides in Metrology, JCGM 100.

[4] Lira I, Grientschnig D. Bayesian assessment of uncertainty in metrology: a tutorial. Metrologia 2010;47:R1–14.

[5] Wübbeler G, Krystek M, Elster C. Evaluation of measurement uncertainty and its numerical calculation by a Monte Carlo method Meas. Sci Technol 2008;19: 084009.

[6] Bich W, Callegaro L, Pennecchi F. Non-linear models and best estimates in the GUM. Metrologia 2006;43:S196–9.

[7] Szczykulska M, Baumgratz T, Datta A. Multi-parameter quantum metrology. Advances in Physics: X 2016;1(4):621–39.

[8] Pierre Squara, Michael Imhoff and Maurizio Cecconi, Metrology in Medicine: From Measurements to Decision, with Specific Reference to Anesthesia and Intensive Care, Anesthesia & Analgesia: January 2015 - Volume 120 - Issue 1 - p 66-75.

[9] Mayer T. Abhandlung über die Umwalzung des Monds um seine Axe und die scheinbare Bewegung der Mondsflecten. In: Kosmographische Nachrichten Und Sammlungen Auf Das Jahr 1748; 1750,. p. 52–183.

[10] Hald A. A history of mathematical statistics from 1750 to 1930. New York: John Wiley; 1998.

[11] Stigler S. The history of statistics: The measure of uncertainty before 1900. Cambridge, MA: Belknap Press and Harvard University Press; 1986.

[12] Wepster SA. Between Theory and Observations Tobias Mayer's Explorations of Lunar Motion. Springer Science+Business Media LLC; 2010. p. 1751–5.

[13] Steele, John M. An Integrated Approach: Tobias Mayer. STEELE, John M. Ancient Astronomical Observations and the Study of the Moon's Motion (1691-1757) [online]. Boston, MA: Springer US, 2012, 2012-12-27, pp. 95-122 [cit. 2021-6-10]. ISBN 978-1-4614-2148-1. doi:10.1007/978-1-4614-2149-8_7.

[14] Eric G. Forbes, The Euler-Mayer Correspondence (1751–1755). London: Macmillan Education UK; 1971.

[15] William Blitzstein, An analysis of the errors in John Flamsteed's Mural Arc observations, Vistas in Astronomy, Volume 41, Issue 1, 1997, Pages 139-155, ISSN 0083-6656, https://doi.org/10.1016/S0083-6656(96)00068-2.

[16] Wallis, D.A. History of Angle Measurement. In: FIG Working Week 2005 and GSDI-8, Cairo, Egypt, 2005, 1-17.

[17] Mekid, S., and Vaja, D. Propagation of uncertainty: Expressions of second and third order uncertainty with third and fourth moments. Measurement 41(6):600-609, 20088.

[18] Hoerl AE, Kennard RW. Ridge regression: Biased eastimation for nonorthogonal problems. Technometrics 1970;12:55–67. https://doi.org/10.1080/00401706.1970.10488634.