TARGET RELIABILITY OF STRUCTURES IN THE CONTEXT OF BAYES' THEOREM

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Abstract
This paper deals with target reliability determination of structures. In the first part, terms risk and reliability are delimited. Further, the difference between the understanding of reliability in civil and machinery engineering is shown. The widely used safety index is a subject of a deeper analysis and its relation to target reliability is presented. The second part of this paper contains a summary of a target reliability estimation, as it is calculated in the present time, and a draft of an alternative approach to the target reliability determination using the Bayes' theorem.

Keywords
risk, safety, structure, probability, Bayes' theorem

1 RISK AND RELIABILITY

The terms of risk and reliability are parts of a quality management terminology. They are basically antagonistic. Whereas risk is widely understood as a combination of frequency or probability of an occurrence of specified dangerous incidents and their consequences, reliability is defined as a stability of utility attributes (functional, ecological, safety, etc.) of a product or an object for certain, preset period of time and under preset conditions of use.

The above-mentioned quality management is (as a part of management in general) hierarchically superior to:

- *Risk management* – focused on identification and analysis of risks, their minimisation, determination of maximum permissible risk, i.e. decision making in order to set an acceptable level of danger while justified public interests are taken into account.
- *Reliability management* – focused on reaching and maintenance of a real risk level on the safe side of the limit level set by risk management and a permanent control of meeting the required criteria.

In the technical point of view, quantitative methods based on theoretic grounds of the theory of probability and statistics are essential. There are two basic ways of quantification used at applications of the mentioned methods, namely:

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- *Measurement* – obtained quantitative signs are variable, i.e. they are subject of a random fluctuation, which shows, however, some internal behavioural patterns. Therefore, it is possible to work with these signs like with a statistically random variable.
- *Numbering* – numbers give the amount of discordant units in a random selection. There are discrete random variables, most often 0 or 1, where only the option yes – no can be declared. These quality signs are called *attributive*.

Traditional (i.e. classic) quantitative statistical methods suitable for this purpose are:

- data files assessment – calculation of basic statistical characteristics and study of obtained histograms,
- use of various graph types designed in order to gain clarity and illustration of studied analyses,
- Pareto analysis – serves to identification of basic decisive reasons of the researched effect. According to Pareto principle, only a few of many possible reasons are significant. It is based on their relative frequency or cumulative relative frequency,
- correlation and regression analysis – for determination or confirmation of stochastic dependency of researched quantities,
- statistic regulation of processes – for statistical stability assessment of researched processes.

2 RELIABILITY OF PRODUCTS AND STRUCTURES

The understanding of reliability problems varies significantly in the structural or machinery engineering point of view. However, at more detailed analysis, reliability in the structural point of view proves to be an organic part of reliability in general.

The seeming contradiction comes in the phase of distinguishing of objects to *repaired* ones and *non-repaired* ones. Basically, it means that:

- *Non-repaired* objects represent the bigger part of factory industrial production nowadays; where product that does not fulfil its function is not more feasible to be repaired and is better to be exchanged for a new one. This applies to e.g. transistors, radios, cookers, etc., as well as to some parts of more complicated machines and instruments (e.g. rotors of electric motors, etc.).
- *Repaired* objects are, for various reasons, not easily replaceable for new ones and are needed to be returned to the working state by repairs, even multiple ones.

Structural objects (structures) belong, obviously, to the repaired set of objects, as it is being confirmed even by the existence and continuous development of an individual discipline of structural production that deals with rehabilitation and reconstruction of structural objects. Only a smaller part of production in structural engineering belongs to the group of non-repaired objects, e.g. prefabrication of structural elements.

3 MEANING OF USED SYMBOLS

Tab. 1 Meaning of used symbols

B	*safety index*
μ_M	*mean value*
σ_M	*standard deviation*
B(p)	*utility from object existence*
C(p)	*price of project and object realization*
C0	*object acquisition price*
C_f	*costs connected to remedy of limit state exceeding*
D(p)	*unknown value*
E	*life expectancy in years*
GDP	*gross domestic product per year and capita*
P_f	*probability of failure*
P_d	*target reliability*
P_{dn}	*probability for n years*
P_{d1}	*probability for 1 year*
P(H)	*probability expression of confidence to H hypothesis before D data receiving*
P(H|D)	*probability expression of confidence to H hypothesis after D data receiving*
R	*structural resistance*
S	*structural loads*
N	*number of years*
P	*vector of all safety parameters*
W	*life share dedicated to economic activity*

4 TARGET RELIABILITY AND SAFETY INDEX

Currently, maximum allowable risk for structures is expressed by two, fully equivalent ways, namely:

- target reliability, marked as P_d in structural engineering, expressing the limit permissible probability of failure P_f,
- safety index, marked as β in structural engineering.

The mutual relation between quantities P_f and β is presented by equation:

$$P_f = \Phi\left(\frac{0-\mu_M}{\sigma_M}\right) = \Phi(-\beta) \quad , \tag{1}$$

where safety requires meeting the condition of:

$$P_f \le P_d \,. \tag{2}$$

Fig. 1 shows the relation of probabilistic and semi-probabilistic approach of safety. The semi-probabilistic layout is expressed by position of the centre of the white triangle, as an unambiguous result. In reality, it is not that simple and the stochastic character of reality is depicted by the set of points. The boundary between satisfactory and unsatisfactory cases creates the line of:

$$R = S\,. \tag{3}$$

The probability density with a graphic interpretation of probability of failure P_f and safety index β can be well seen in the cross-section.

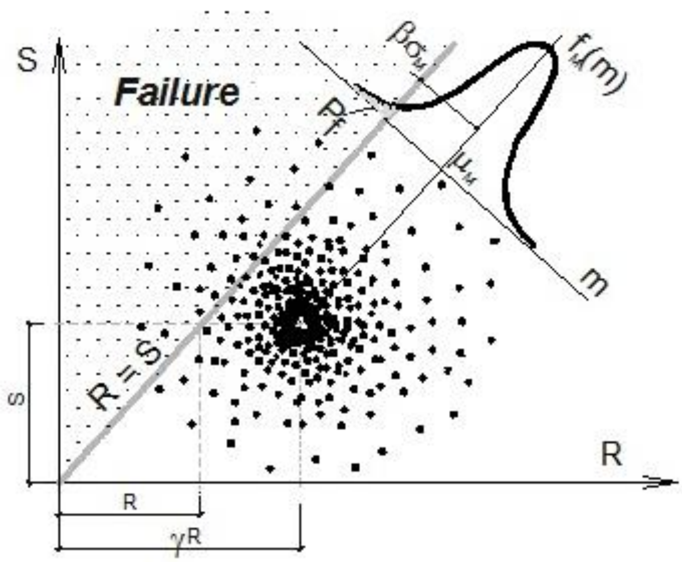

Fig. 1 Relations of various structural safety theories [14, 16]

As expressed in formula (1), quantities P_f and β are fully equivalent. Their calculation is accomplished by methods of statistics and probability. There are many proven procedures of these calculations.

On the contrary, the value of P_d determines the target reliability that is acceptable in the limit states point of view. The determination of this value does not fully comply with the probability characterisation of the structures reliability problems and is burdened, in many cases, by semi-probabilistic way of thinking that is valid in contemporary standards.

5 TARGET RELIABILITY DETERMINATION

Key values of *R* and *S* functions have been determined by deterministic, later semi-probabilistic methods. Currently, a complete shift to fully probabilistic solutions of formulae (2) and (3) is on the way. Simulation methods based on applications of Monte Carlo method are, along with an outstanding development of computers, mighty means to universal implementation of this quality turn.

In the deterministic approach, it is sufficient if the unambiguous solution of formula (3) is located in the safe area $R > S$, in the worst case on the boundary $R = S$. The probabilistic approach is ambiguous, as shown by the set of points on fig. 1. It admits the $R < S$ cases happen inevitably, too. As the reliability and safety measure, a permissible presumable number of these unfavourable cases are accepted.

For the probabilistic solution, formula (2) transforms into formula:

$$P_f = P[R - S < 0] \leq P_d \, . \tag{4}$$

Considering the simulation methods as well elaborated, the problem of probabilistic control of system reliability reduces to a responsible determination of maximum permissible limit of P_d.

Obviously, it is not possible to set one universal limit value of P_d. There are many issues that influence the choice of P_d, e.g.: system operation conditions, external natural forces, severity of consequences of system breakdown, etc. Moreover, it is not completely clear whether an exceptional value of P_d shall be set for exceptional design situations. More frequent catastrophic situations as a result of terrorist attacks and global warming of the Earth, in particular, provide good reasons to this move. However, the only different values of P_d are for ultimate limit states and service limit states.

Examples of exceptional P_d values set for exceptional design situations are following:

- probability of failure of nuclear power plant is set in the USA as $P_d = 10^{-6}$,
- probability of crash of an airliner within one hour of flight is set as $P_d = 10^{-9}$.

Values of ordinary design situations are following:

- $10^{-7} < P_d < 10^{-3}$ for ultimate limit states,
- $10^{-3} < P_d < 10^{-1}$ for service limit states.

Recently, a shift towards service limit states design could be clearly noticed. This approach better characterizes the influences of corrosion and external environment in general as they are deciding in the long-term reliability point of view. The SBRA simulation method shows [1], based on structure importance and limit state group:

Tab. 2 SBRA values P_d

Structure Importance	Ultimate Limit State	Service Limit State
Low	0.000 5	0.16
Average	0.000 07	0.07
High	0.000 008	0.023

Another approach comes from the probability of failure in an interval of one year; for determination of probability of failure in a longer interval, the formula stands as:

$$P_{dn} = 1 - (1 - P_{d1})^n \, , \tag{5}$$

where n is number of years.

The economic point of view respects relation:

$$P_d \approx \frac{C_0}{C_f} 10^{-3} \, . \tag{6}$$

Apparently, the higher the cost connected with remedy of exceeding of the limit state C_f is, the lower the permissible P_d is.

JCCS recommendation [14] takes into account economic viewpoints, too. Preliminary values of P_f and β with respect to the interval of 1 year and the considered limit state are presented in the tables 3 and 4. It is apt to notice that in accordance with our conventions and meaning of eq. (2), P_d appears as a more correct notation in these tables instead of P_f.

Tab. 3 Limit values for interval of 1 year and ultimate limit states

Reconstruction Costs	Low Failure Consequences	Average Failure Consequences	High Failure Consequences
High	$\beta = 3.1$ ($P_f = 10^{-3}$)	$\beta = 3.3$ ($P_f = 10^{-4}$)	$\beta = 3.7$ ($P_f = 10^{-4}$)
Normal	$\beta = 3.7$ ($P_f = 10^{-4}$)	$\beta = 4.2$ ($P_f = 10^{-5}$)	$\beta = 4.4$ ($P_f = 10^{-5}$)
Low	$\beta = 4.2$ ($P_f = 10^{-5}$)	$\beta = 4.4$ ($P_f = 10^{-6}$)	$\beta = 4.7$ ($P_f = 10^{-6}$)

Tab. 4 Limit values for 1 year interval and service limit states

Reconstruction Costs	Limit Values
High	$\beta = 1.3$ ($P_f = 10^{-1}$)
Average	$\beta = 1.7$ ($P_f = 5.10^{-2}$)
Low	$\beta = 2.3$ ($P_f = 10^{-2}$)

It is clear to see that there is not a unifying principle in such important problems like determination of the target reliability of building structures. This results in new tendencies to find a satisfactory theoretical outcome that enables determining of P_d values.

Contemporary efforts aim to define so called optimal and best practice structure, which use "economically considered" best practice design.

Regardless the theoretical details (published e.g. in [15, 16]), the determination of optimum value of the target reliability leads to consequences in human life loss expressed by LQI (Life Quality Index). LQI is a conglomerate of social indicators and is characterized by definition:

$$LQI = GDP^{w} E^{1-w} . \tag{7}$$

i. e. that life quality and its value, too, are assessed according to purchasing power of population. This is, apparently, different across different countries in the world. [15] presents a general relation of life expectancy E depending on purchasing power and its graphic presentation is shown on Fig. 2.

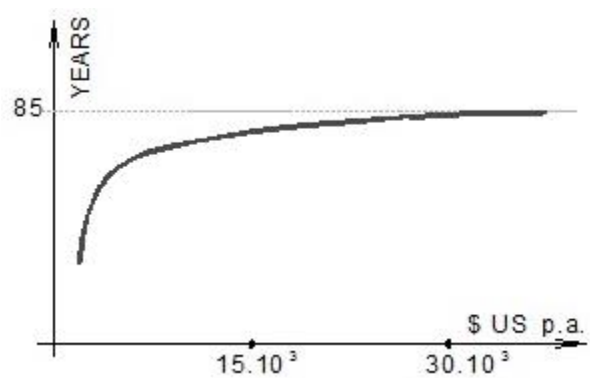

Fig. 2 Curve of life expectancy depending on yearly purchasing power of population [14, 16]

The relation is well defined by the equation:

$$E = a \ln(P - b) + c , \tag{8}$$

with following values of parameters:

a = 7.1874,
b = 371.5,
c = 6.2075,
P = stands here for yearly purchasing power (horizontal graph axis of Fig. 2):

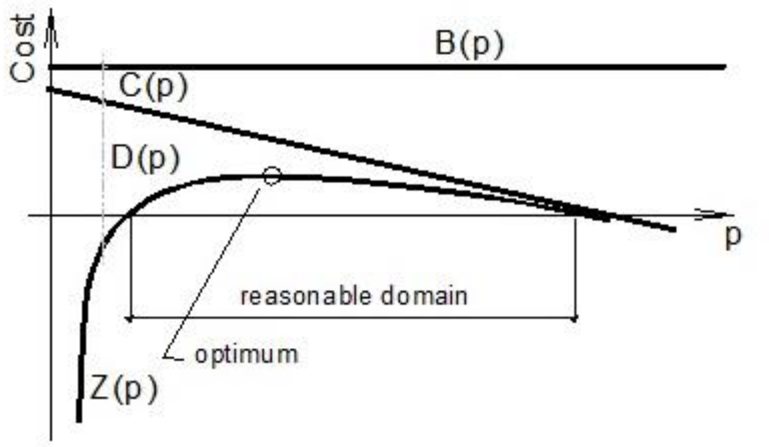

Fig. 3 Progress of the objective function Z(p) – equation (9), [14, 16]

[16] assumes that LQI can be considered as a social indicator for establishing a theory of target reliability and technical objects, including structures, optimization, too. Target reliability of a structure can be considered as optimal, according to [16], if the following objective function is maximized (see Fig. 3):

$$Z(p) = B(p) - C(p) - D(p)\,, \tag{9}$$

All quantities in equation (9) are measured in monetary units.

All other contemplations suppose that the complete equation (9) is differentiable by *p*. At the same time, *C(p)* is considered as rising, whereas *D(p)* as sinking and *B(p)* as constant. Only positive values of *Z(p)* are suitable for reliability calculations – hence target reliability calculation, too. “Suitable“ modifications of objective functions for various reliability situations are considered at the same time. Elaborated mathematical theory is indicated and accessible in [11, 15, 16].

6 BAYES‘ STATISTICS POSSIBILITIES

The foregoing text indicates that a target reliability value is valid for a certain group of objects (structures) for given limit state if the target reliability is proposed (by any of the above presented method or value).

It is desired in the economic point of view that not all the individual objects are made on the same quality level, and target reliability value can be changed at every single object based on regular observations (rechecks – measurements). Regrettably saying, building objects are generally not controlled that much over time to make the advantages of Bayes‘ statistics usable.

Bayes‘ theory of probability, called subjective probability, too, provides theoretical apparatus to specification of a priori probability based on running control. It delivers a postulate that:

A priori probability emerging from information before an experiment has been done can be modified to a posterior probability after one or more experiments that are random have been done. A posterior probability respects both initial knowledge and selective information from additionally made experiments.

Bayes‘ formula was firstly published in 1763 in the posthumously edited work of Presbyterian ecclesiastic Thomas R. Bayes: An Essay Towards Solving a Problem in the Doctrine of Chances. Formula, easily derivable with rules of conditional probability, caused confusion and disputes among statisticians, mathematicians and philosophers for coming 242 years (i.e. until present time).

Bayes formula in its most simplified form for hypothesis *H* and data *D* can be expressed in the following way:

$$P(H \mid D) = \frac{P(D \mid H)P(H)}{P(D)}, \tag{10}$$

Instructions for given *P(D|H)* and *P(D)* how to learn from gained data are contained in the theorem.

Bayes‘ theorem is published in different (as needed) variations for practical calculations, e.g.:

- For getting a new information about commence of A effect at conditional probability of B_j, *j=1,2,…n, effects:*

$$P(B_j \mid A) = \frac{P(B_j)P(A \mid B_j)}{P(A)}. \tag{11}$$

- For random selection of *n* range with intentions to study a random quantity X_j, it is possible to set an a posterior probability density *θ* based on selection observations *x = (xj), j = 1,2…n:*

$$f(\theta \mid x) = \frac{f(\theta)L(x \mid \theta)}{\int f(\theta)L(x \mid \theta)d\theta}. \tag{12}$$

where $L(x|\theta) = f(x_j)$ is so called likelihood function and integration in the denominator applies to all possible *θ* values.

7 EXAMPLE OF BAYES’ THEOREM APPLICATION IN CIVIL ENGINEERING

Civil engineering structures are sometimes located on unbearable subsoil. Typical way of foundation procedure in unbearable subsoil is drilling of piles. By drilling of a pile, these failures can, apart from others, occur: when drilling a pile in gravel, a larger boulder or piece of rock can prevent the pile from penetration; when drilling a pile through clay, it is important not to stop drilling until the pile is fully placed, otherwise the clay can stick to the pile and significantly higher force (sometimes more than drilling machine is able to create) is necessary to set it into move again [17].

In an example, take a civil engineering company facing a task to build foundations of a bridge on a certain place with unbearable subsoil. The geological survey has shown, in the area of interest, 50 % of the subsoil is created by gravel, 40 % by sand, and 10 % by clays. If the pile failed to be placed into its final position and the company does not have any other information about probability of failures of this kind, it can assume that there was probability of 50 % that the pile failure was caused by gravel, 40 % by sand, and 10 % by clay. However, if the company was collecting information about the probabilities of failure of pile drilling (e.g. from own research, from other reliable data, etc.), it can estimate the probabilities more precisely using the Bayes’ theorem.

Let us suppose, in compliance with the previous paragraph, event A1: The material in the expected position of drilling of a pile is gravel. The corresponding probability is P(A1) = 0.5. Event A2: Sand, P(A2) = 0.4. Event A3: Clay, P(A3) = 0.1. Event B: The pile fails to get to its final position. The company can have additional information that probability of pile failure in gravel (e.g. due to hitting a larger piece of rock) is 5 %, and in clay 40 % (e.g. due to possible stop and inability to move again). Sands are very well drillable and let us assume the probability of pile failure is 0.1 %. These values correspond to P(B|A1) = 0.05, P(B|A2) = 0.001, and P(B|A3) = 0.4. Total probability of the pile failure is then:

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3), \tag{13}$$

which results in P(B) = 0.0654. Calculation of probability of the pile failure using Bayes' theorem (11) results in P(A1|B) = 0.382, P(A2|B) = 0.006, and P(A3|B) = 0.612. These results show very different perspective on failure source than the results prior to the incorporation of the additional information and the use of Bayes' theorem. Although clay is expected only in 10 % of pile locations, when a pile failed to drill into its final position, there is 61.2 % probability it occurred due to clay. More examples on the use of the Bayes' theorem can be found at [18].

8 CONCLUSION

Methods of probability assessment of structural objects – structures – are theoretically widely elaborated. Calculated results are compared to limit values, here referred to as target reliability. Any strict laws do not control determination procedures of these limits and there is rather a period of seeking after these laws. The key is in answer to this fundamental question: *What target reliabilities are safe enough?*

New information about risk level can be obtained by the use of Bayes' theorem independently on the way the limit value was set. This would, however, mean a regular control of important qualitative (essential for safety) indicators of finished "repaired" work. This does not happen in structural engineering, yet.

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