

The Magnitude-Optimum Design of the PI Controller for Plants with Dead Time and Oscillatory Modes

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Abstract: The design of the PI controller for stable linear plants based on the magnitude-optimum criterion can be applied directly for high-order linear models with dead time, without need of the model reduction. This paper discusses key properties of this method, especially considering oscillating plants. Even though the closed-loop response is specified only for low frequencies and the robustness requirements are not explicitly formulated in the design objective, in the case of plants without zeros a large closed-loop stability margin is ensured automatically regardless the dead time value, under the assumption that the damping coefficients corresponding to the oscillatory modes of the plant are sufficiently high.

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Keywords: dead time, time delay, frequency response, magnitude optimum, PI controller, PID controller, process control, stability margin.

1. INTRODUCTION

The analytical design of the PID controller for linear plants is complicated by dead time, which is present in models of many industrial processes. Most practical tuning methods are based on simplified low-order models and utilize an approximation of the dead time dynamics. A direct controller design for higher-order plants with time delays, not using a model reduction, enables to achieve enhanced performance, but is usually much more complex. One purely analytical tuning approach is based on the magnitude optimum or modulus optimum (MO) criterion – Åström and Hägglund (1995), Vrančić et al. (1999), Cvejn (2013), which requires that the magnitude of the closed-loop frequency response between the plant output and the reference signal is as flat as possible in the low-frequency band. Denote $G_{CL}(s)$ the closed-loop transfer function between the reference signal and the plant output. Considering $\lim_{\omega \rightarrow 0} |G_{CL}(i\omega)| = 1$, the mentioned requirement can be formally written as

$$\lim_{\omega \rightarrow 0} \frac{d^k}{d\omega^k} |G_{CL}(i\omega)| = 0, \quad k = 1, 2, \dots, k_m \quad (1)$$

where k_m is as high as possible. The MO criterion is most suitable for the reference tracking control tasks, where the closed-loop system is to be able to respond quickly to changes of the reference input, or to efficiently reject the disturbances affecting the plant output directly. To improve the performance of rejection of the plant-input disturbances a modification of the MO criterion has been proposed in Vrančić et al. (2004, 2010). In Cvejn and Vrančić (2018) the MO-tuned controller is extended with a suitable first-order term to enhance the disturbance rejection capabilities, while preserving the closed-loop stability margin properties of the MO method.

Since the closed-loop response is prescribed only for low frequencies, it can be expected that the control loop behavior for middle frequencies, decisive for the closed-loop stability, need not be convenient in general. Nevertheless, it turns out that in "normal" situations

$$L(\omega) \in \{z \mid \operatorname{Re} z \geq -0.5\}, \quad \forall \omega \geq 0 \quad (2)$$

holds, where $L(\omega) \stackrel{\text{def}}{=} R(i\omega)F(i\omega)$ denotes the open-loop frequency response function. This property guarantees $M_s \leq 2$, where $M_s = \sup_{\omega \geq 0} |1 + L(\omega)|^{-1}$ is the closed-loop sensitivity, which is recommended for the PID controller tuning in general – see Åström and Hägglund (1995). The property (2) also ensures the gain margin greater than 2 and the phase margin greater than 60° . On the other hand, the method can fail to produce stabilizing settings for stable plants in some cases, or can give settings with a reduced stability margin. This especially regards plants with zeros, but the problem can occur in the case of plants with complex roots as well.

The properties of the MO tuning method for the PI controller and stable plants with the transfer function in the form

$$F(s) = \frac{K}{a_n s^n + \dots + a_1 s + 1} e^{-\tau s} \quad (3)$$

have been studied in Cvejn (2022). Although the controller has only two tuning parameters, it turns out that the problem is rather complex. A goal of this paper is to present these results in a simplified and more concise form. The PI controller transfer function is considered in the form

$$R(s) = K^{-1} (r_0 + r_{-1}/s) \quad (4)$$

where K is the plant static gain in (3). In this way the parameter K is excluded from the manipulations below.

2. COMPUTATION OF THE MO SETTINGS

Consider the Taylor expansion of $F(s)$ at $s = 0$ in the form

$$F(s) = K(1 - c_1s + c_2s^2 - c_3s^3 + \dots) \tag{5}$$

Then it is possible to obtain

$$\begin{aligned} \operatorname{Re} L(\omega) &= K^{-1} [r_0 \operatorname{Re} F(i\omega) + r_{-1} \operatorname{Im} F(i\omega) / \omega] = \\ &= (r_0 - c_1r_{-1}) + (-c_2r_0 + c_3r_{-1})\omega^2 + \dots \end{aligned} \tag{6}$$

The requirement (1) equivalent to

$$\lim_{\omega \rightarrow 0} \frac{d^k}{d\omega^k} [1 + 2 \operatorname{Re} L(\omega)] = 0, \quad k = 0, \dots, k_m \tag{7}$$

for the plants (3), where equations (7) for odd k are satisfied automatically – see, e.g., Cvejn (2013). Substituting (6) into (7) gives the system of linear equations

$$r_0 - c_1r_{-1} = -0.5, \quad -c_2r_0 + c_3r_{-1} = 0 \tag{8}$$

for the controller settings. Based on Vrančić et al. (1995), the coefficients c_k can be computed directly from the coefficients of the transfer function (3) as follows:

$$\begin{aligned} c_1 &= a_1 + \tau \\ \dots & \\ c_k &= (-1)^{k+1} a_k + \frac{\tau^k}{k!} + \sum_{i=1}^{k-1} (-1)^{k+i-1} c_i a_{k-i} \end{aligned} \tag{9}$$

3. THE MO SETTINGS FOR $F(s)$ IN FACTORIZED FORM

Although the MO settings can be computed directly from (8) and (9), the model (3) is not suitable for studying the stability properties of the method. For these purposes, $F(s)$ is considered in the factorized form

$$F(s) = K \prod_{k=1}^m (\alpha_k T_k^2 s^2 + T_k s + 1)^{-1} e^{-\tau s} \tag{10}$$

where $m \leq n$, $T_k > 0$ and $\alpha_k \geq 0$. If $\alpha_k > 1/4$ for a particular k , the corresponding factor in the denominator

$$\alpha_k T_k^2 s^2 + T_k s + 1 = (\sqrt{\alpha_k} T_k)^2 s^2 + 2\zeta_k (\sqrt{\alpha_k} T_k) s + 1 \tag{11}$$

has a pair of complex roots with the damping ratio $\zeta_k = 1 / (2\sqrt{\alpha_k})$, whereas $\alpha_k \in (0, 1/4]$ corresponds to a pair of real roots. The case $\alpha_k = 0$ corresponds to a single real root. Note that $m = n$ in (3) if $\alpha_k = 0$ for all k , otherwise $m < n$.

Denote

$$G(\omega) = \frac{|F(i\omega)|}{K} = \left[\prod_{k=1}^m (1 - \alpha_k (T_k \omega)^2)^2 + T_k^2 \omega^2 \right]^{-1/2}, \tag{12}$$

$$H(\omega) = -\angle F(i\omega) = \tau\omega + \sum_{k=1}^m \operatorname{atan}_2(T_k \omega, 1 - \alpha_k (T_k \omega)^2) \tag{13}$$

where the function $\operatorname{atan}_2(y, x)$ returns φ in the interval $(-\pi, \pi]$ such that $r \sin \varphi = y$ and $r \cos \varphi = x$, $r > 0$.

It is useful to make the following transformation of the frequency. If we define the dimensionless frequency $\xi = T_\Sigma \omega$, where

$$T_\Sigma = \sum_{k=1}^m T_k + \tau \tag{14}$$

and denote $g(\xi) = G(\xi/T_\Sigma)$ and $h(\xi) = H(\xi/T_\Sigma)$, the open-loop frequency response function can be expressed in terms of ξ as $L(\omega) = l(\xi)$, where

$$l(\xi) = g(\xi) \left(r_0 + T_\Sigma \frac{r_{-1}}{i\xi} \right) e^{-ih(\xi)}. \tag{15}$$

The following result is obtained in Cvejn (2022).

Table 1. The coefficients β_{jk} values in dependence on α_k

j	β_{jk}
1	1
2	$1 - 2\alpha_k$
3	$1 - 3\alpha_k$
4	$1 - 4\alpha_k + 2\alpha_k^2$
5	$1 - 5\alpha_k + 5\alpha_k^2$

Proposition 1. Let

$$s_j = T_\Sigma^{-j} \sum_{k=1}^m \beta_{jk} T_k^j \tag{16}$$

where the coefficients β_{jk} are defined in Table 1 in dependence on α_k . The functions $g(\xi)$ and $h(\xi)$ can be expanded as

$$g(\xi) = 1 + g_2 \xi^2 + g_4 \xi^4 + \dots, \quad h(\xi) = \xi + h_3 \xi^3 + \dots \tag{17}$$

where $h_{2j+1} = (-1)^j s_{2j+1} / (2j+1)$ for $j \geq 1$, and

$$g_2 = -\frac{s_2}{2}, \quad g_4 = \frac{s_4}{4} + \frac{s_2^2}{8}, \quad g_6 = -\left(\frac{s_6}{6} + \frac{s_2 s_4}{8} + \frac{s_2^3}{48} \right). \tag{18}$$

Figure 1 shows the values of β_{jk} in dependence on α_k in graphical form. Note that for plants with only real poles all β_{jk} s equal 1.

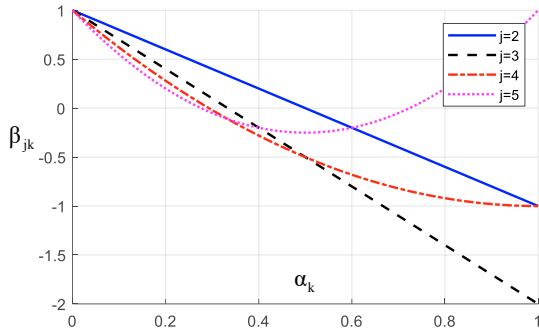


Figure 1. The values of β_{jk} in dependence on α_k .

Substituting (17) into $F(i\omega) = Kg(\xi)e^{-ih(\xi)}$, expanding into the form (5), and solving the system of equations (8) yields

$$r_{-1} = \frac{3}{4T_\Sigma} \frac{1+s_2}{1-s_3} = \frac{3}{4} \frac{T_\Sigma^2 + \sum_{k=1}^m (1-2\alpha_k)T_k^2}{T_\Sigma^3 - \sum_{k=1}^m (1-3\alpha_k)T_k^3}, \quad (19)$$

$$r_0 = \frac{3}{4} \frac{1+s_2}{1-s_3} - \frac{1}{2} = T_\Sigma r_{-1} - \frac{1}{2}. \quad (20)$$

4. THE STABILITY MARGIN OF THE MO TUNING METHOD

4.1. General frequency-domain properties of the MO settings

From (7) it follows that $\text{Re}L(\omega) \rightarrow -0.5$ for $\omega \rightarrow 0$. Since the frequency transformation $\xi = T_\Sigma \omega$ does not affect the shape of the Nyquist plot, for analyzing properties of the MO settings it is possible to work with $l(\xi)$ instead of $L(\omega)$.

Denote ξ_u the lowest transformed frequency such that $\angle l(\xi) = -\pi$. If $\text{Re}l(\xi)$ is increasing for $\xi \in [0, \xi_u]$ and $|l(\xi)|$ is non-increasing for $\xi \geq \xi_u$, it is sure that $\text{Re}l(\xi) \geq -0.5$ for all $\xi \geq 0$, so the closed-loop stability is ensured by the Nyquist criterion, and $M_s \leq 2$ holds.

However, note that $|l(\xi)|' < 0$ for $\xi \in [0, \xi_u]$ does not necessarily mean that $\text{Re}l(\xi)$ is increasing in this interval.

Let $A(\xi) = \sin h(\xi)/\xi$, $B(\xi) = \cos h(\xi)$, and

$$\sigma = \frac{r_0}{T_\Sigma r_{-1}} = 1 - \frac{2(1-s_3)}{3(1+s_2)}. \quad (21)$$

Then $\text{Re}l(\xi) = -T_\Sigma r_{-1} V(\xi)$, where

$$V(\xi) = g(\xi)[A(\xi) - \sigma B(\xi)]. \quad (22)$$

If $s_3 < 1$ and $s_2 > -1$, $r_{-1} > 0$ is always obtained by substitution into (19), which is necessary for the closed-loop

stability, as discussed above. The requirement $s_3 < 1$ ensures bounded values of r_0 and r_{-1} . It is easily seen from (21) that $r_{-1} > 0$ is equivalent to $\sigma < 1$ and $\sigma \geq 0$ is equivalent to $r_0 \geq 0$.

4.2. The trend of $\text{Re}l(\xi)$ for $\xi \rightarrow 0$

Since $l(\xi)$ is close to the line $\{z | \text{Re}z = -0.5\}$ for low ξ due to (7), decreasing trend of $\text{Re}l(\xi)$ signalizes that the stability margin gets reduced. Therefore, non-decreasing trend of $\text{Re}l(\xi)$ for $\xi \rightarrow 0$ is a basic indicator of proper open-loop behavior for the MO settings. Since $\sigma < 1$ is assumed, it is $V(0) > 0$, and increasing $\text{Re}l(\xi)$ corresponds to decreasing $V(\xi)$.

Considering $V(\xi) = v_0 + v_2 \xi^2 + v_4 \xi^4 + \dots$, $v_2 = 0$ holds due to the MO-optimality. Therefore, $V(\xi)$ is decreasing for $\xi \rightarrow 0$ if $v_4 < 0$. By substituting the expansion of $h(\xi)$ into the Taylor expansions of $\cos h(\xi)$ and $\sin h(\xi)$, and by multiplying with (17), it is possible to obtain that the requirement $v_4 < 0$ is equivalent to

$$\frac{1-\sigma}{8} (2s_4 - s_2^2) - \left(\frac{1}{24} + \frac{s_3}{3} \right) \sigma + \frac{s_5}{5} + \frac{s_3}{6} + \frac{1}{120} < 0. \quad (23)$$

It is not difficult to verify that the inequality (23) holds if s_j are non-negative, $s_3 < 1$, and $s_{j+1} \leq s_j^{(j+1)/j}$ for $j = 2, \dots, 4$. If $\beta_{jk} = 1$ for all $k, j \geq 1$, these inequalities are equivalent to

$$\sum_{k=1}^m z_k^p \leq \left(\sum_{k=1}^m z_k \right)^p \quad (24)$$

where $z_k = (T_k / T_\Sigma)^j$ and $p = (j+1)/j$, which is the triangle inequality in L_p -space. This situation corresponds to the plants (10) with only real roots, where $s_j \geq 0$ and $s_3 < 1$ are fulfilled if $n > 1$ or $\tau > 0$. However, in the case of complex roots the conditions $s_{j+1} \leq s_j^{(j+1)/j}$ are often not satisfied.

Therefore, the factors $\alpha_k T_k^2 s^2 + T_k s + 1$ in the plant (10) are divided into two groups: group A contains all the factors such that $\beta_{3k} \geq 0$, i.e. $\alpha_k \leq 1/3$, and group B contains the factors where $\beta_{3k} < 0$. Denote I_A and I_B the sets of indices k corresponding to the group A and B , respectively, so $I_A \cup I_B = \{1, \dots, m\}$ and $I_A \cap I_B = \emptyset$. The terms s_j are thus decomposed as $s_j = s_j^A + s_j^B$, where for $x_k = T_k / T_\Sigma$ it is

$$s_j^A = \sum_{k \in I_A} \beta_{jk} x_k^j, \quad s_j^B = \sum_{k \in I_B} \beta_{jk} x_k^j. \quad (25)$$

For $j \leq 3$ and $l = 1, 2$ it can be verified that $\beta_{j+l,k} \leq \beta_{jk}^{(j+l)/j}$ for $\alpha_k \leq 1/3$. Consequently, $s_{j+l}^A \leq (s_j^A)^{(j+l)/j}$ holds. In addition, if $\alpha_k \leq 0.5$, it is $s_2^B \geq 0$ and $s_j^B \leq 0$, $j = 3, \dots, 5$. Therefore, $s_5 \leq (s_3^A)^{5/3}$ and $s_4 \leq (s_3^A)^{4/3}$, which means that the inequality (23) is satisfied if

$$\begin{aligned} \psi(z) \stackrel{\text{def}}{=} & (1-\sigma) \frac{2(s_3^A)^{4/3} - s_2^2}{8} - \left(\frac{1}{24} + \frac{s_3}{3} \right) \sigma + \\ & + \frac{(s_3^A)^{5/3}}{5} + \frac{s_3}{6} + \frac{1}{120} < 0 \end{aligned} \quad (26)$$

where $s_2 = s_2^A + s_2^B$, $s_3 = s_3^A + s_3^B$, σ is given by (21), and z denotes a point in the set containing all the feasible configurations described by the values of s_2^A , s_2^B , s_3^A and s_3^B .

By definition $s_1^A + s_1^B \leq 1$, and due to (24) it is $s_2^A \leq (s_1^A)^2$ and $s_3^A \leq (s_2^A)^{3/2}$. Therefore, $s_1^B \leq 1 - \eta$ and $s_3^A \leq \eta^3$, where $\eta = (s_2^A)^{1/2}$. If $\alpha_k \leq 0.5$ for all k , it is $s_2^B \leq (s_1^B)^2/3$ and $s_3^B \geq -0.5(s_1^B)^3$, because $\beta_{2k} \leq 1/3$ and $\beta_{3k} \geq -0.5$ for $k \in I_B$ (see Tab. 1 and Fig. 1). Consequently, it is sufficient to verify that $\psi(z) < 0$ for all points $z = (\eta, s_2^B, s_3^A, s_3^B)$, where $\eta \in [0, 1]$, $s_2^A = \eta^2$, and

$$s_2^B \in \left[0, \frac{(1-\eta)^2}{3} \right], s_3^A \in [0, \eta^3], s_3^B \in [-0.5(1-\eta)^3, 0]. \quad (27)$$

Validity of $\psi(z) < 0$ on the compact set defined by the system of inequalities (27) can be easily verified with help of computer - see Cvejn (2022) for details. It is possible to formulate the following result.

Proposition 2. Considering the plant (10) and the controller (4), $\text{Re}l(\xi)$ is increasing for $\xi \rightarrow 0$ if $\alpha_k \leq 0.5$ in (10) for all $k = 1, \dots, m$, and $n > 1$ or $\tau > 0$.

The requirement $\alpha_k \leq 0.5$ in Proposition 2 corresponds to $\zeta_k \geq 1/\sqrt{2}$. For higher values of α_k the intervals of s_2^B and s_3^B have to be extended and $\psi(z) < 0$ in Σ no longer holds. In addition, since β_{3k} is increasing for $\alpha_k > 0.5$ and is positive for $\alpha_k > 0.72$, the validity of (23) is not guaranteed by $\psi(z) < 0$ due to the term s_5^B , which is no longer negative.

Although $\alpha_k > 0.5$ for some k in (10) does not necessarily mean that the stability margin is reduced, in some such cases $\text{Re}l(\xi)$ may be decreasing for $\xi \rightarrow 0$ even if $|l(\xi)|$ is

decreasing for all $\xi \geq 0$. In addition, if $\alpha_k \geq 1$ for some k , $r_{-1} < 0$ may result, which implies closed-loop instability. For instance, for the plant

$$F_{\alpha,T}(s) = 1 / (\alpha T^2 s^2 + Ts + 1) \quad (28)$$

it is $s_2 = -1$ for $\alpha = 1$, which corresponds to $r_{-1} = 0$ and $\sigma \rightarrow -\infty$. The closed-loop system not stable if $\alpha > 1$.

4.3. The behavior of $l(\xi)$ for middle and high frequencies

If $\alpha_k \in [0, 0.5]$ for all k , the requirement $|l(\xi)|' \leq 0$ for $\xi \geq \xi_u$ is always satisfied, because $g(\xi)$ is monotone, hence $|l(\xi)| = T_\Sigma r_{-1} g(\xi) |\sigma - i/\xi|$ is non-increasing. If there are factors in $F(s)$ such that $\alpha_k > 0.5$, $g(\xi)$ can be increasing for $\xi \geq \xi_u$ and $|l(\xi)|' \leq 0$ need not hold in general.

Even if $\text{Re}l(\xi)$ is increasing for $\xi \rightarrow 0$ and $|l(\xi)|' \leq 0$ for all $\xi \geq 0$, the trend of $\text{Re}l(\xi)$ should be inspected for higher ξ as well, especially in a neighborhood of the open-loop ultimate frequency ξ_u . However, this analysis seems to be rather complicated and can be only outlined here.

It is a common property of all stable plants (10) that $h(\xi) > 0$ is increasing. In addition, if $\sigma < 1$, it is $\lim_{h \rightarrow \infty} h(\xi) \geq \pi$. This allows to construct the inverse function to $h(\xi)$, denoted $\xi(h)$ in the sequel, which is increasing and bounded in any interval $[0, h_1]$, where $h_1 \in [0, \pi)$. Denote $g_h(h) = g(\xi(h))$ and

$$A_h(h) = \sin h / \xi(h), B_h(h) = \cos h. \quad (29)$$

Since $\angle l(\xi) < -\pi$ for $h(\xi) = \pi$, it is $h(\xi_u) < \pi$. The function $\text{Re}l(\xi)$ is non-decreasing for $\xi \in [0, \xi_u]$ if $V_h(h) \stackrel{\text{def}}{=} g_h(h) V_{1h}(h)$ is non-increasing for $h \in [0, \pi]$, where $V_{1h}(h) \stackrel{\text{def}}{=} A_h(h) - \sigma B_h(h)$.

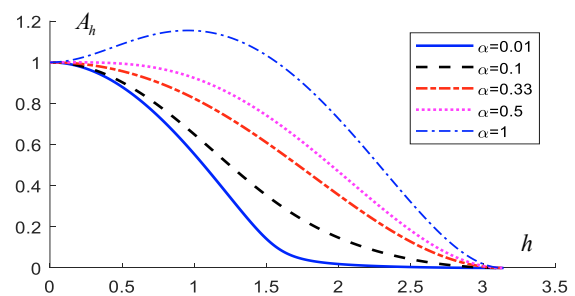


Figure 2. The plots of $A_h(h)$ for the plants $F_{\alpha,T}(s)$ with $T = 1$ and different α .

Considering $\alpha_k \in [0, 0.5]$, the function $A_h(h)$ is decreasing and has a single flex point, denoted h_A^* , which appears when $A_h(h)$ gets sufficiently close to zero. Fig. 2 shows the plots of $A_h(h)$ for several plants $F_{\alpha,T}(s)$ (28). To estimate the minimal value of h_A^* , it seems to be sufficient to inspect only the family of plants $F_{\alpha,T}(s)$ for different values of α , because additional factors in the denominator of $F(s)$ or dead time can only increase h_A^* . The minimal value of h_A^* is about 1.15 and corresponds to $\alpha \approx 0.1$.

Further, since $V_{1h}(0) = 1 - \sigma > 0$ and $V_{1h}(\pi) = \sigma$, $V_{1h}(h) > 0$ always holds for sufficiently low h . If $V_{1h}(h) \leq 0$ for higher h , it is also $V_h(h) \leq 0$, and in such a case $V_h(h) < V_h(0)$, so the stability margin (2) cannot be violated. Therefore it is possible to consider only $V_{1h}(h) > 0$. Since $g_h(h) > 0$, the requirement $V_h'(h) \leq 0$ is equivalent to

$$\gamma(h)V_{1h}(h) + V_{1h}'(h) \leq 0 \quad (30)$$

where $\gamma(h) \stackrel{\text{def}}{=} g_h'(h)/g_h(h) = (\ln g_h(h))'$. If $\alpha_k \in [0, 0.5]$, the function $\gamma(h)$ is always decreasing in the interval $[0, \pi/2]$. Since $\gamma(h) < 0$, the MO optimality and (30) imply $V_{1h}'(h) \geq 0$ for low h . Since $V_{1h}'(h) < 0$ means that $V_h'(h) < 0$, it is possible to consider $V_{1h}'(h) \geq 0$ in the whole interval $[0, \pi/2]$. The condition (30) can be then written as

$$\Lambda(h) \stackrel{\text{def}}{=} -\gamma(h)V_{1h}(h) + \sigma B_h'(h) \geq A_h'(h) \quad (31)$$

where the first term in $\Lambda(h)$ is increasing. For $h \rightarrow \pi/2$ it is $B_h''(h) = 0$, so $\Lambda(h)$ must be increasing in the upper part of $[0, \pi/2]$. Since $A_h'(h)$ is decreasing for $h \leq h_A^*$, $\Lambda(h) \geq A_h'(h)$ is preserved in $[0, \pi/2]$ if $h_A^* \geq \pi/2$.

If $h_A^* < \pi/2$, $B_h'(h)$ is nearly constant in $[h_A^*, \pi/2]$, because h_A^* is close to $\pi/2$, so both $A_h'(h)$ and $\Lambda(h)$ are increasing in $[h_A^*, \pi/2]$. It is possible to explain that if $h_A^* < \pi/2$, $A_h''(h)$ is monotone in $[h_A^*, \pi/2]$ in the case of plants (10) – see Cvejn (2022). Consider the expansions

$$A_h'(h_A^* + \delta) \approx \alpha_0 + \alpha_2 \delta^2, \quad \Lambda(h_A^* + \delta) \approx \beta_0 + \beta_1 \delta + \beta_2 \delta^2 \quad (32)$$

where $\alpha_0 < \beta_0$, since $A_h'(h_A^*) < \Lambda(h_A^*)$, as discussed above, and $\beta_1 > 0$. From (32) it can be seen that if $A_h'(h) = \Lambda(h)$ for some $h \in (h_A^*, \pi/2)$, $A_h'(\pi/2) > \Lambda(\pi/2)$ holds. These

arguments explain that, considering $V_h'(h) \leq 0$ for low h , it is sufficient to inspect the trend of $V_h'(h)$ only for $h \geq \pi/2$.

On the other hand, it is possible to show that for $h \in [\pi/2, \pi)$, $V_h'(h)$ of any plant $F(s)$ in the form (10) is always lower than $V_h'(h)$ corresponding to some plant

$$\hat{F}(s) = F_{\alpha,T}(s)e^{-\hat{\tau}s} \quad (33)$$

where $F_{\alpha,T}(s)$ is given by (28) – see Cvejn (2022). Due to this fact, it is sufficient to verify that $\hat{V}_h'(h) \leq 0$ in the interval $[\pi/2, \pi)$. It turns out that only $\alpha \in [0, 0.5]$ can be inspected. Denote $\hat{\xi}(h)$, $\hat{g}(h)$ and $\hat{V}_h(h)$ the functions $\xi(h)$, $g(h)$ and $V_h(h)$ corresponding to $\hat{F}(s)$.

Although the validity of $\hat{V}_h'(h) \leq 0$ for $h \in [\pi/2, \pi)$ has not been proven analytically, the functions $\hat{\xi}(h)$ and $\hat{g}(h)$ depend only on two parameters: α and $\mathcal{G} = T/(T + \hat{\tau})$, besides the argument h . This enables easy verification by means of computer. The requirement $\hat{V}_h'(h) < 0$ can be written as

$$\theta_{\alpha,\mathcal{G}}(h) \stackrel{\text{def}}{=} \gamma(h) \left(\frac{\sin h}{\hat{\xi}(h)} - \hat{\sigma} \cos h \right) + \frac{\cos h}{\hat{\xi}(h)} + \left(\hat{\sigma} - \hat{\xi}'(h)/\hat{\xi}(h)^2 \right) \sin h < 0 \quad (34)$$

where the value of $\hat{\xi}(h)$ can be for $h \in [\pi/2, \pi)$ obtained iteratively by bisection, and $\hat{\xi}'(h) = 1/\hat{h}'(\hat{\xi}(h))$.

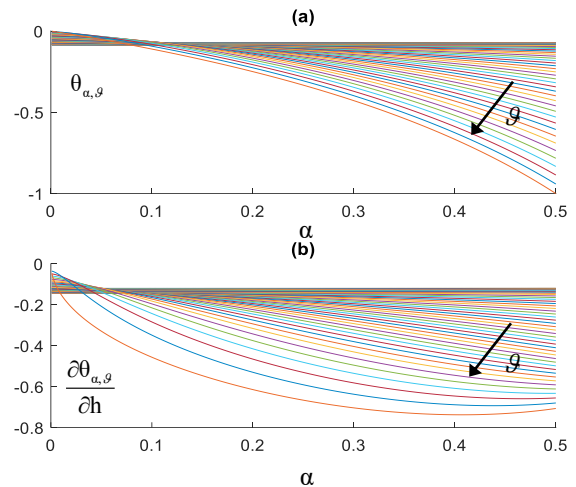


Figure 3. The values of (a) $\theta_{\alpha,\mathcal{G}}(\pi/2)$ and (b) $\partial\theta_{\alpha,\mathcal{G}}/\partial h$ at $h = \pi/2$, for discrete steps of \mathcal{G} in $[0, 1]$ and variable $\alpha \in [0, 0.5]$.

Fig. 3a shows the plots of $\theta_{\alpha,g}(\pi/2)$ in dependence on $\alpha \in [0, 0.5]$, for discrete steps of $g \in [0, 1]$. It can be seen that $\theta_{\alpha,g}(\pi/2) \rightarrow 0$ only if $\alpha \rightarrow 0$ and $g \rightarrow 1$. In addition, Fig. 3b shows the corresponding plots of $\partial\theta_{\alpha,g}/\partial h|_{h=\pi/2}$ obtained by numerical differentiation. Since $\partial\theta/\partial h < 0$, $\theta_{\alpha,g}(h)$ tends to decrease with respect to h for $h \geq \pi/2$.

5. SIMULATED RESULTS AND CONCLUSIONS

The properties of the MO tuning method for the PI controller are demonstrated on the family of plants in the form

$$F_k(s) = \frac{e^{-\tau s}}{(T_0 s + 1)(\alpha T^2 s^2 + T s + 1)} \quad (35)$$

for different values of the parameters T_0 , T , α and τ . Fig. 4 shows the MO-optimal Nyquist plots and corresponding closed-loop reference signal step responses. The controller settings are computed by using the formulas (19)-(20).

Table 2. The plants $F_k(s)$ parameters

k	T_0	T	α	τ
1	0.5	1	0.2	0
2	0.5	1	0.5	0.5
3	0	1	0.8	2
4	0	1	0.8	0
5	1	0.1	1	0

The plants $F_1(s)$ and $F_2(s)$ are well damped, and with a moderate value of τ . The responses are not oscillating and have a similar overshoot of about 7%. The plants $F_3(s)$ and $F_4(s)$ contain a low-damped mode. In addition, $F_3(s)$ has dominant dead time dynamics. Surprisingly, the dead time speeds up the response in this case. In the case of $F_4(s)$ a negative r_0 results, which causes an undershoot slowing down the response. The case $F_5(s)$ is the most peculiar. Since T is much lower than T_0 , it could be expected that the complex term had a minor influence. However, the stability margin is strongly reduced. This problem cannot occur if $\alpha_k \leq 0.5$ for all $k = 1, \dots, m$, as explained in Section 4.2. Overall, the dead time τ has a positive effect on the stability margin, which seems to be a characteristic feature of this tuning method.

In comparison to other approaches, the MO settings seem to be preferable for well damped plants with important dead time dynamics. A simulation-based comparison with the AMIGO method, which confirms this, can be found in Cvejn (2022). However, oscillatory modes with $\zeta_k < 1/\sqrt{2}$ can cause inverse response or a reduced stability margin, even if $G(\omega)$ is monotone. In the case of plants with zeros,

especially the LHP zeros often lead to strongly reduced stability margin or even closed-loop instability. The corresponding analysis, however, is not available at current.

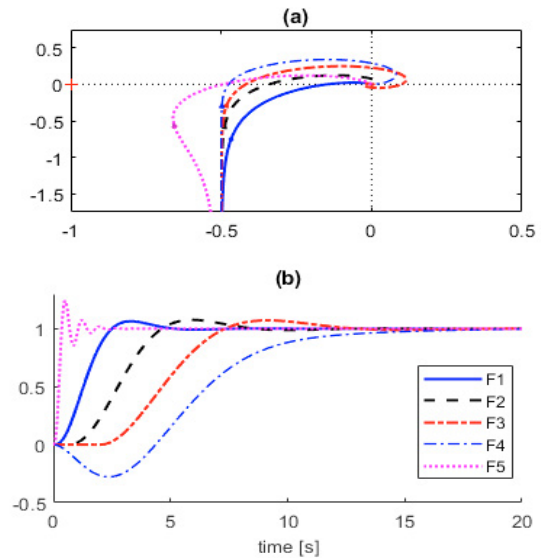


Figure 4. The MO method: (a) the open-loop Nyquist plots and (b) the corresponding closed-loop reference signal step responses for the plants $F_k(s)$.

REFERENCES

- Åström, K.J., Hägglund, T. (1995). *PID Controllers: Theory, Design, and Tuning*. 2nd edition. Instrument Society of America, USA.
- Cvejn, J., Vrančić, D. (2018). The magnitude optimum tuning of the PID controller: Improving load disturbance rejection by extending the controller. *Transactions of the Institute of Measurement and Control*, 40(5), 1669-1680.
- Cvejn, J. (2013). The design of PID controller for non-oscillating time-delayed plants with guaranteed stability margin based on the modulus optimum criterion. *Journal of Process Control*, 23(4), 570-584.
- Cvejn, J. (2022). The magnitude optimum design of the PI controller for plants with complex roots and dead time. *Archives of Control Sciences*, 32(1), 5–35.
- Vrančić, D., Peng, Y., and Strmčnik, S. (1999). A new PID controller tuning method based on multiple integrations. *Control Engineering Practice*, 7(5), 623-633.
- Vrančić, D., Strmčnik, S., Kocijan, J. (2004). Improving disturbance rejection of PI controllers by means of the magnitude optimum method. *ISA Transactions*, 43(1), 73-84.
- Vrančić, D., Strmčnik, S., Kocijan, J., and Oliviera, P.M. (2010). Improving disturbance rejection of PID controllers by means of the magnitude optimum method. *ISA Transactions*, 49(1), 47–56.