Pension-related application of the cohort life table.

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Abstract: *Longevity risk, the risk that people will live longer than expected, weighs heavily on those who run pension schemes and on insurers that provide annuities. Hence the prediction of future mortality rates is an issue of fundamental importance for the insurance and pensions industry.*
Our analysis focuses on mortality at higher ages (65-95), given our interest in pension-related applications where the risk associated with longer-term cash flow is primarily linked to uncertainty in future rates of mortality. The Lee-Carter model became one of the most applied models and it is used to forecast age-specific death rates. The main goal of this paper is to apply the Lee-Carter model to construct the so-called "cohort life tables" for calculation of a 30-year annuity to a person aged 65 in 2015.
We use data on deaths and exposures for the Czech Republic from the Human Mortality Database (HMD). The HMD provides evidence that life expectancy is increasing. We have shown that if the today rate of increase will continue, it will at age 65 concluded (after calculation) to increase the present value of pension liabilities in defined-benefit schemes about 5 % if we use cohort life table instead of period life table.
Probability statements derived from the use of a single model and parameter set should be treated with caution. Hence, there is a need for awareness of model risk when assessing longevity-related liabilities.

Key words: longevity risk, annuity, stochastic mortality, life table, Lee-Carter model,

JEL Classifications: C53, G22, J11, J32

1 Introduction

Benjamin Franklin said: "In this world nothing can be said certain, except death and taxes." The death is certain, but the timing is much less certain.

The mortality of the population in developed countries has improved rapidly over the last thirty years and this has important financial implications for the insurance industry, since several important classes of liability are sensitive to the direction of future mortality trends. This uncertainty about the future development of mortality gives rise to **longevity risk**.

Longevity risk, the risk that people live longer than expected, weighs heavily on those who run pension schemes and on insurers that provide annuities. The risk that the reserves established for the payment of benefits (retirement, widowhood, orphan hood, disability, dependency,…) are inadequate for that purpose if they are based on life tables (or mortality tables) with lower survival hypothesis than real. Longevity risk plays a central role in the insurance company management since only careful assumptions about future evolution of mortality phenomenon allow the company to correctly face its future obligations. Longevity risk represents a sub-modul of the underwriting risk module in the Solvency II framework. The most popular and widely used model for projecting longevity is the well-known Lee-Carter model.

This paper follows on articles Gogola, J. (2014), Gogola, J. (2014a), Gogola, J. (2015), Jindrová, P., Slavíček, O. (2012), Pacáková, V., Jindrová, P. (2014) and Pacáková, V., Jindrová, P., Seinerová, K. (2013). They deal with the development and the prediction of life expectancy in selected European countries (Czech Republic, Slovakia, Finland and

Spain) by applying Lee-Carter model and the Quantification of Selected Factors of Longevity.

Most stochastic mortality models are constructed in a similar manner. Specifically, when they are fitted to historical data, one or more time-varying parameters (κ_t) are identified. By extrapolating these parameters to the future, we can obtain a forecast of future death probabilities and consequently other demographic quantities such as life expectancies. They are important for quantifying longevity in pension risks and for constructing benchmarks for longevity-linked liabilities. The main goal of this paper is to apply the Lee-Carter model to construct the so-called "cohort life tables" and use them for calculation of a 30-year annuity to a person aged 65 in 2015.

2 Methodology and Data

We use data of the total population, males and females' deaths and exposure to risk between 1950 and 2014 for the Czech Republic (CR) from the Human Mortality Database (www.mortality.org). We consider the restricted age range from 0 to 95.

Let calendar year *t* runs from exact time *t* to exact time *t*+1 and let $d_{x,t}$ be the number of deaths aged *x* last birthday in the calendar year *t*. We suppose that the data on deaths are arranged in a matrix $\boldsymbol{D} = (d_{x,t})$. In a similar way, the data on exposure are arranged in a matrix $\boldsymbol{E}^c = (e_{x,t})$ where $e_{x,t}$ is a measure of the average population size aged *x* last birthday in calendar year *t*, the so-called central exposed to risk. We suppose that $(d_{x,t})$ and $(e_{x,t})$ are each $n_a \times n_y$ matrices, so that we have n_a ages and n_y years.

We denote the *force of mortality* (or *hazard rate*) at exact time *t* for lives with exact age *x* by $\mu_{x,t}$. The force of mortality can be thought as an instantaneous death rate, the probability that a life subject to a force of mortality $\mu_{x,t}$ dies in the interval of time $(t, t+\mathrm{d}t)$ is approximately $\mu_{x,t} \cdot \mathrm{d}t$ where $\mathrm{d}t$ is small.

The force of mortality $\mu_{x,t}$ for human populations varies slowly in both *x* and *t* and a standard assumption is that $\mu_{x,t}$ is constant over each year of age, i.e., from exact age *x* to exact age *x*+1, and over each calendar year, i.e., from exact time *t* to exact time *t*+1. Thus,

$$\mu_{x+u,t+v} = \mu_{x,t} \text{ for } 0 \le u < 1, 0 \le v < 1, \tag{1}$$

and so $\mu_{x,t}$ approximate the mid-year force of mortality $\mu_{x+0.5,t+0.5}$.

We suppose that $d_{x,t}$ is a realization of a Poisson variable $D_{x,t}$:

$$D_{x,t} \sim Po(e_{x,t} \cdot \mu_{x,t}), \tag{2}$$

The expected values are the product of exposures $e_{x,t}$ and the force of mortality $\mu_{x,t}$.

Assumption (2) leads us to the maximum likelihood estimates of $\mu_{x,t}^{\mathrm{MLE}} = m_{x,t}$ as

$$m_{x,t} = \frac{d_{x,t}}{e_{x,t}}, \tag{3}$$

or in a matrix form $m = \dfrac{\boldsymbol{D}}{\boldsymbol{E}^c}$, that means element-wise division in **R**.

We also consider the mortality rate $q_{x,t}$. This is the probability that an individual aged exactly *x* at exact time *t* will die between *t* and *t* +1.

We have the following relation between the force of mortality and the mortality rate:

$$q_{x,t} = 1 - \exp\left(\int_0^1 -\mu_{x+s,t+s} \, \mathrm{ds}\right) \approx 1 - e^{-\mu_{x,t}} . \quad (4)$$

We use the following conventions for our model:

- The α_x, β_x coefficients will reflect age-related effects
- The κ_t coefficients will reflect time-related effects

Our models are fitting to historical data.

The Lee-Carter model was introduced by Ronald D. Lee and Lawrence Carter in 1992 with the article Lee, R. D., Carter, L. (1992). The model grew out of their work in the late 1980s and early 1990s attempting to use inverse projection to infer rates in historical demography. The model has been used by the United States Social Security Administration, the US Census Bureau and the United Nations. It has become the most widely used mortality forecasting technique in the world today.

Lee and Carter proposed the following model for the force of mortality:

$$\log m_{x,t} = \alpha_x + \beta_x \cdot \kappa_t \ , \quad (5)$$

with constraints

$$\sum_{x=1}^{n_a} \beta_x = 1 , \quad (6)$$

$$\sum_{t=1}^{n_y} \kappa_t = 0 . \quad (7)$$

The second constraint implies that, for each *x*, the estimate for α_x will be equal (at least approximately) to the mean over *t* of the $\log m_{x,t}$.

Let ϕ represent the full set of a parameters and the notation for $\mu_{x,t}$ is extended to $\mu_{x,t}(\phi)$, to indicate its dependence on these parameters.

For our model the log-likelihood is:

$$l(\phi; \boldsymbol{D}, \boldsymbol{E}) = \sum_{x,t} \left(d_{x,t} \cdot \log[e_{x,t} \cdot \mu_{x,t}(\phi)] - e_{x,t} \cdot \mu_{x,t}(\phi) - \log(d_{x,t}!) \right), \quad (8)$$

and estimation is by maximum likelihood (MLE).

By the equation (5) the log of the force mortality is expressed as the sum of an age-specific component α_x that is independent of time and another component that is the product of a time-varying parameter κ_t reflecting the general level of mortality and an age-specific component β_x that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes.

Interpretation of the parameters in Lee-Carter model is quite simple: $exp(\alpha_x)$ is the general shape of the mortality schedule and the actual forces of mortality change according to overall mortality index κ_t modulated by an age response β_x (the shape of the β_x profile tells which rates decline rapidly and which slowly over time in response of change in κ_t).

For practice the fitting of a model is usually only the first step and the main purpose is the forecasting of mortality. For forecasting-time series we use Random Walk with Drift.

The estimated age parameters, α_x, β_x, are assumed invariant over time. This last assumption is certainly an approximation but the method has been very thoroughly tested in Booth, H., Tickle, L., Smith, L. (2005) and found to work.
We assume that trend observed in past years can be graduated (or smoothed) and that it will continue in future years.
By the Random Walk with Drift the dynamics of κ_t follows

$$\kappa_t = \kappa_{t-1} + \theta + \varepsilon_{t-1}, \quad (9)$$

with i.i.d standard Gaussian distribution $\varepsilon_t \sim \mathrm{N}(0; \sigma_\varepsilon^2)$.
Value at future time $t+h$ can be written as

$$\kappa_{t+h} = \kappa_t + h \cdot \theta + \sum_{s=0}^{h-1} \varepsilon_{t+s}, \quad (10)$$

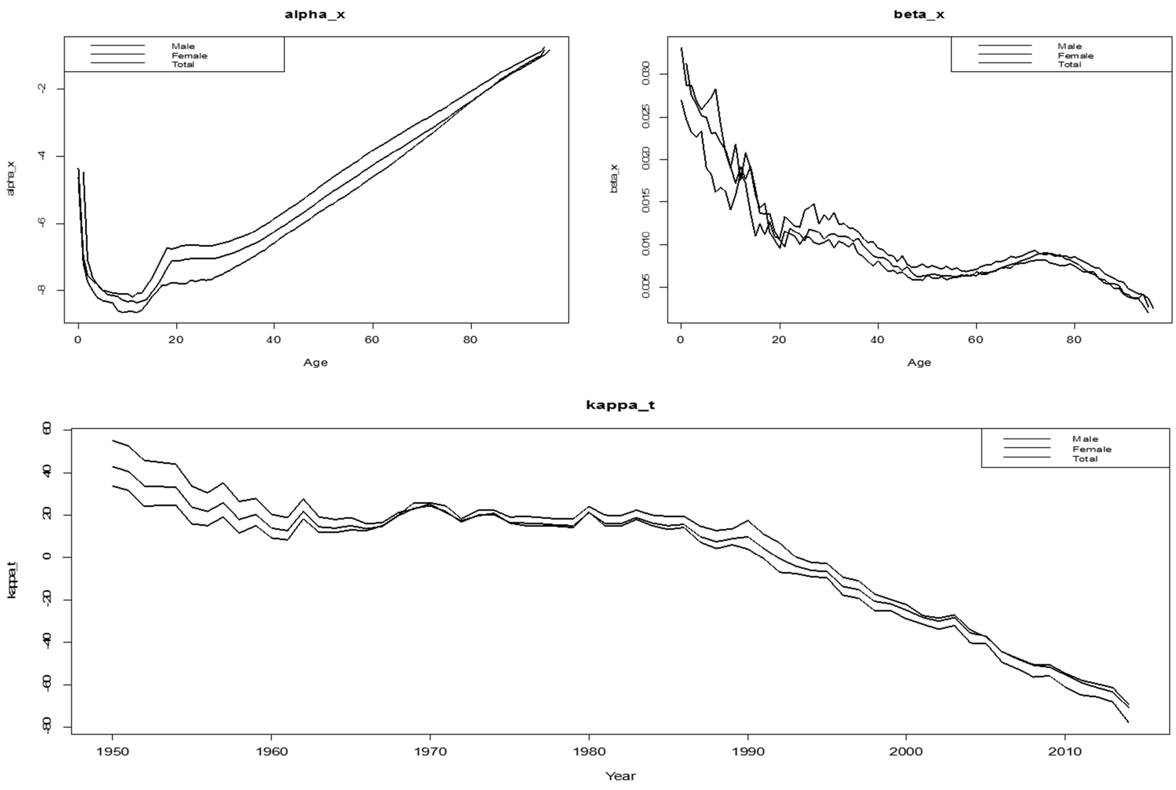
which has Gaussian distribution $\mathrm{N}(\kappa_t + h \cdot \theta; \sigma_\varepsilon^2 \cdot h)$.
Hence the best point estimate for future value at time $t+h$ is $\kappa_t + h \cdot \theta$,
and the 95% confident interval is

$$(\kappa_t + h \cdot \theta - 1{,}96 \cdot \sigma_\varepsilon \cdot \sqrt{h}; \kappa_t + h \cdot \theta + 1{,}96 \cdot \sigma_\varepsilon \cdot \sqrt{h}). \quad (11)$$

3 Results

In Figure 1 we have plotted the maximum likelihood estimates for the parameters of the Lee-Carter model, using CR total population data, aged 0-95. Model fitting was done in **R**, which was also used for graphs (Figure 1). Note that estimated values for β_x are higher at the lowest ages, meaning that at those ages the mortality improvements are faster. The decreasing trend in κ_t reflects general improvements in mortality over time at all ages.

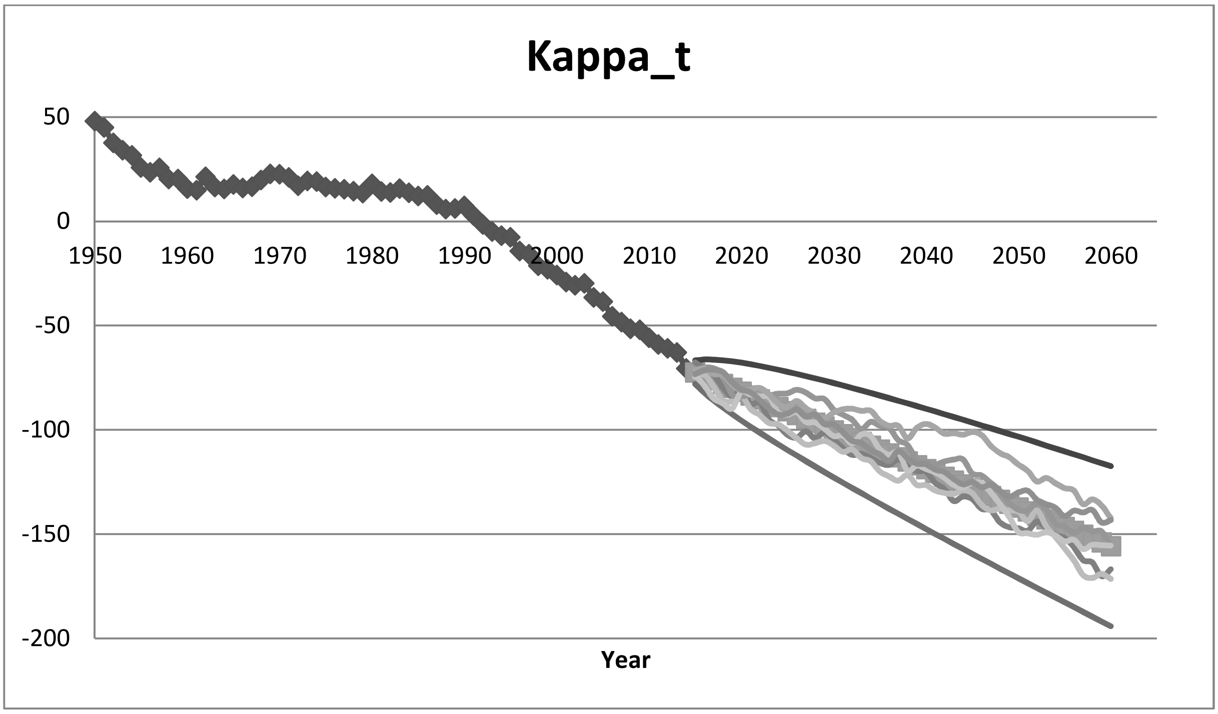
Figure 1 Estimated parameters of the Lee-Carter model for population of CR



Source: Own processing

We will now simulate the κ_t up to 2060 according to equation (9). These results in case of the total population are plotted in Figure 2. The dashed curves in plot show the 2,5-th and 97,5-th percentile of the distribution of κ_t resulting in a 95 % confidence interval. By forecasted κ_t we get the predictions for the force of mortality $\mu_{x,t} = \exp(\alpha_x + \beta_x \cdot \kappa_t)$, which lead us by equation (4) to mortality rates $q_{x,t}$.

Figure 2 Predicted κ_t for total population with 95 % CI



Source: Own processing

To avoid underestimation of the relevant liabilities we use dynamic mortality model. Cohort or dynamic life table provide a view on the future evolution of mortality rates and it implies the diagonal arrangement in a projecting life table (see Table 1).

Table 1 Period life table vs. Cohort life table

$q_{x,t}$	2014	2015	2016	2017	2018	2019	2020
.	.	.	.	.	.	.	.
65	0.014699	0.014505	0.014314	0.014125	0.013938	0.013754	0.013573
66	0.015832	0.015618	0.015406	0.015197	0.014991	0.014788	0.014587
67	0.017191	0.016954	0.016721	0.016491	0.016263	0.016039	0.015818
68	0.018574	0.018311	0.018051	0.017795	0.017543	0.017294	0.017048
69	0.020037	0.019744	0.019456	0.019172	0.018892	0.018615	0.018343
70	0.021675	0.02135	0.021029	0.020714	0.020403	0.020097	0.019795
71	0.023349	0.02299	0.022637	0.022289	0.021946	0.021609	0.021276
.	.	.	.	.	.	.	.

Source: Own calculations

Finally by equations (12)-(15) we find the present values of the annuities such as term immediate annuity $a_{x:\bar{n}|}$, term annuity-due $\ddot{a}_{x:\bar{n}|}$. We will also consider annuities payable *m*-times per year.

$$a_{x:\bar{n}|} = \sum_{t=1}^{n} v^t \cdot \ {}_t p_x \quad (12)$$

$$a_{x:\bar{n}|}^{(m)} = a_{x:\bar{n}|} + \frac{m-1}{2m} \cdot (1 - v^n \cdot \ {}_n p_x) \qquad \text{(UDD)} \quad (13)$$

$$\ddot{a}_{x:\bar{n}|} = \sum_{t=0}^{n-1} v^t \cdot \ {}_t p_x \quad (14)$$

$$\ddot{a}_{x:\bar{n}|}^{(m)} = \ddot{a}_{x:\bar{n}|} - \frac{m-1}{2m} \cdot (1 - v^n \cdot \ {}_n p_x) \qquad \text{(UDD)} \quad (15)$$

(where (UDD) means the assumption of Uniform Distribution of Deaths).

Take an individual aged 65 in 2015 (birth year = 1950) who wants to purchase a 30 years annuity. For calculation annuities first we use the Period table, which contains the last available mortality rates. In our case it is year 2014 (the second column of Table 1). Then we use the diagonal values (Cohort table) for the cohort aged 65 in 2015 (born 1950) who are still alive in year 2015+t.

Table 2 gives present values of 30 years annuities for the individual aged 65 from the whole population of the Czech Republic with interest rate of 2 % p.a.
In appendix (Table 3 and Table 4) we show present values of annuities separately for both genders.

Table 2 Present values of annuities for the total population in the Czech Republic (x=65, n=30, i=0,02)

	$a_{x:\bar{n}|}$	$a_{x:\bar{n}|}^{(12)}$	$\ddot{a}_{x:\bar{n}|}^{(12)}$	$\ddot{a}_{x:\bar{n}|}$
Period table	14.04	14.31	14.74	15.01
Cohort table	14.75	15.01	15.43	15.69
Relative change	**5.01%**	**4.85%**	**4.69%**	**4.54%**
2.5%	14.13	14.40	14.83	15.09
	0.65%	0.61%	0.60%	0.57%
97.5%	15.34	15.60	16.01	16.27
	9.26%	8.99%	8.64%	8.39%

Source: Own calculations

4 Conclusions

National governments and the WHO announce life expectancies of different populations every year. To financial institutions, life expectancy is not an adequate measure of risk, because all it does not give any idea about how mortality rates at different ages vary over time. On the other hand, indicators of longevity risk cannot be too complicated. An indicator that is composed by a huge array of numbers is difficult to interpret and will lose the purpose as a "summary" of a mortality pattern.
We have presented stochastic models to analyse the mortality and shown how they may be fitted. Afterwards we can turn to the industry requirement to forecast future mortality. We have shown that if the today rate of increase will continue, it will at age 65 concluded (after calculation) to increase the present value of pension liabilities in defined-benefit schemes cca. 4,5-5 % if we use cohort life table instead of period life table.
But forecasting of mortality should be approached with both caution and humility. Any prediction is unlikely to be correct.
There is a need for awareness of model risk when assessing longevity-related liabilities, especially for annuities and pensions. The fact that parameters can be estimated does not imply that they can sensibly be forecast.
Such forecasting should enable actuaries to examine the financial consequences with different models and hence to come to an informed assessment of the impact of longevity risk on the portfolios in their care.

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Appendix

Table 3 Present values of annuities for males (x=65, n=30, i=0,02) in the CR

	$a_{x:\bar{n}\|}$	$a^{(12)}_{x:\bar{n}\|}$	$\ddot{a}^{(12)}_{x:\bar{n}\|}$	$\ddot{a}_{x:\bar{n}\|}$
Period table	12.68	12.95	13.38	13.66
Cohort table	13.29	13.55	13.99	14.25
Relative change	**4.83%**	**4.65%**	**4.54%**	**4.38%**
2.5%	12.72	12.99	13.43	13.70
	0.37%	0.33%	0.36%	0.32%
97.5%	13.85	14.11	14.54	14.80
	9.26%	8.95%	8.69%	8.40%

Source: Own calculations

Table 4 Present values of annuities for females (x=65, n=30, i=0,02) in the CR

	$a_{x:\bar{n}\vert}$	$a_{x:\bar{n}\vert}^{(12)}$	$\ddot{a}_{x:\bar{n}\vert}^{(12)}$	$\ddot{a}_{x:\bar{n}\vert}$
Period table	15.14	15.41	15.83	16.10
Cohort table	15.92	16.18	16.58	16.84
Relative change	**5.13%**	**4.98%**	**4.78%**	**4.64%**
2.5%	15.27	15.53	15.95	16.21
	0.83%	0.79%	0.76%	0.72%
97.5%	16.54	16.79	17.18	17.44
	9.23%	8.97%	8.59%	8.36%

Source: Own calculations