NONLINEAR WHEEL/RAIL CONTACT GEOMETRY
CHARACTERISTICS & DETERMINATION OF HERTZIAN CONTACT

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Abstract
Railway vehicle running behavior is mainly determined by the forces that exist between wheel and rail surfaces. These forces are influenced by the nonlinear wheel/rail contact. Furthermore, for better organization of condition monitoring and repair of railway vehicles, better understanding of wheel/rail contact is essential. In this study, analysis of wheel/rail geometrical properties is given along with a detailed explanation of search of contact points and solution of normal problem. The commonly used wheel/rail pair, S1002-UIC60E1, is given as an example. The methodology given in this work is same as the methodology followed by most of the commercial software for dynamical analysis of vehicles.

1. INTRODUCTION
As dynamics of railway vehicles is mainly influenced by wheel/rail contact, understanding of wheel/rail contact is conditio sine qua non to have faster and safer vehicles. Furthermore, it is known that wheel/rail contact is essential for manufacturing and repair technologies as these two technologies are directly related to faster and safer vehicles.

In wheel/rail contact search yaw motion of the wheelset can be ignored as it does not have much influence on contact during the motion of wheelset on a straight track with
In this analysis of wheel/rail contact, the single point contact is given as a consequence of considering new theoretical S1002 and UIC60E1 rail profiles. Furthermore, the deformation of wheel/rail is also considered by using the quasi-elastic contact search method. The yaw motion is not included in this study, whereas the roll motion of the wheelset is included. As wheelset shifts laterally on the track, especially when contact with flange occurs, considerable roll angle appears and it changes the contact point. In order to calculate coordinates of contact points the analytical calculation method is given in (2) and (3). In (4) a semi-analytical method to determine contact locus is given based on geometrical analysis which is also explained in this study. This method is more computationally efficient than finding analytical solution of a set of nonlinear equations.

After seminal work of Hertz (5), his theory is used for the calculation of pressure distribution on the contact area and solution of the wheel/rail normal problem. For fast calculation, approximation functions are proposed based on the tabular data consisting the coefficients given by Hertz (6), (7), (8). These coefficients are found by solving the elliptical integrals given by Hertz. Such a table which is more detailed can be found in (9). Whereas in this case of study, numerical solution for elliptical integrals will be given.

2. WHEEL/RAIL GEOMETRY

In this study, tabular data of wheel/rail surfaces are built up based on technical drawings of the S1002 type wheel (10) and UIC60E1 type rail (11). It should be noted that rail are imbedded in track with an inclination. The value of 1:40 is used here – the profile data are rotated about the track gauge distance measurement point by using the well-known Euler’s rotation theorem. Track gauge is taken as 1435 mm in this work.

\[ \tan \gamma_l = \frac{dz_l}{dy_l}, \quad \tan \gamma_r = \frac{dz_r}{dy_r} \]  

(1)

Fig. 1 Characteristics of wheel and rail

Moreover, also conicity angles for wheel and rail must be found, as this characteristic is especially important about finding the components of contact forces. The conicity angle is defined as
where $\gamma_l, \gamma_r$ are conicity angle of left and right contact points. It should be noted that the first derivatives are needed in order to find conicity angles. In this case numerical differentiation method of first order given by (12) is used. Results shown in Fig. 1 are validated by comparing the results found in (3)

3. RIGID CONTACT SEARCH & SEMI-ANALYTICAL METHOD for CONTACT LOCUS

In this section, wheel/rail contact parameters are given as the functions of only lateral displacement. Semi-analytical method given by (4) is used, for details of the method readers are referred to read that study. Rigid contact search method is based on the finding minimum distance on the z axis. Condition for the contact point is

$$\min d(y, z) = 0,$$  \hspace{1cm} (2)

where $d$ represents the distance between wheel/rail profiles. In other words rigid contact search is a numerical method based on finding the zeros of the distance function given in Eq. 2. Semi-analytical method given by (4) is based on the fact that the vertical distances at the contact points of right and left wheels are equal. As emphasized in Eq. 2 first step is to find vertical distance and calculate the respective minima $d_{\text{min}_r}, d_{\text{min}_l}$ for each point. If these two points are contact points then these minima are equal. If they are not equal roll angle must be adjusted. Supposing $d_{\text{min}_r} > d_{\text{min}_l}$, then the wheelset must be rotated clockwise with an angle

$$\Delta \theta = \frac{d_{\text{min}_r} - d_{\text{min}_l}}{y_{\text{min}_r} - y_{\text{min}_l}}.$$  \hspace{1cm} (3)

Rotation must be repeated until distances for two contact points are equal. Equality is satisfied in terms of tolerance $\varepsilon$. This means

$$|d_{\text{min}_r} - d_{\text{min}_l}| < \varepsilon.$$  \hspace{1cm} (4)

For engineering calculations $10^{-4}$ can be considered as enough tolerance. Finally roll angle can be calculated as

$$\theta = \theta_0 + \sum_{i=1}^{k} \Delta \theta_i.$$  \hspace{1cm} (5)

![Fig. 2](image)

**Fig. 2** Characteristics related to lateral movement of wheelset
where \( k \) is the number of iterations and \( \theta_0 \) is the initial roll angle. In this work initial roll angle is taken as zero for all points. In Fig. 2 several characteristics obtained by lateral movement of wheelset are given. The last subfigure given in Fig. 2 represents the coefficient of contact angle difference parameter which the calculation is given in (13). The results obtained are validated by comparing the results given in (13).

4. QUASI-ELASTIC CONTACT SEARCH

Elastic contact search method is a more realistic approach, but it requires usage of finite element methods and computational complexity is high. Hereby, in this study, the quasi-elastic method, which is also given in (14), (15), (16), is presented. This method has lower computational complexity than elastic contact search and is more accurate than rigid contact search. Fig. 3 demonstrates the parameters used for quasi-elastic method.

In here \( s = y - y_c \), where \( y_c \) is the coordinate of the rigid contact point on \( y \) axis, \( d(s,z_{\text{wheel}},z_{\text{rail}}) \) is the vertical distance function between the wheel and rail surfaces. This method uses the weighted averaging of the distance function in the area of the contact patch. The maximum deformation occurs at the contact patch and for other points become distant, deformation decreases. The assumption here is that the relationship between deformations of both surfaces is exponential. The weight function with respect to the distance function can be given as

\[
w(s,z_{\text{wheel}},z_{\text{rail}}) = \exp \left( \frac{-d(s,z_{\text{wheel}},z_{\text{rail}})}{\varepsilon} \right),
\]

where \( \varepsilon \) is the regularization parameter stated in (15). It should be noted that the points far from contact patch are negligible as weight function increases with an increasing vertical distance. The regularization parameter is chosen so that vertical displacement of the wheel has same size as the elastic deformation in pure elastic normal contact model (16). For the selected S1002 wheel and UIC60E1 rail profiles it is in the range of \( 10^{-5} \cdots 5 \times 10^{-5} \). In calculations of this work it is taken as \( 10^{-5} \). The contact location in terms of \( s \) is given as

**Fig. 3 Wheel/rail contact model and definitions used for quasi-elastic contact search**

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\[
\bar{s} = \frac{\int_{s_{\text{min}}}^{s_{\text{max}}} s w(s, z_{\text{wheel}}, z_{\text{rail}}) ds}{\int_{s_{\text{min}}}^{s_{\text{max}}} w(s, z_{\text{wheel}}, z_{\text{rail}}) ds}.
\] (7)

Therefore \( \bar{s} \) is the location of the contact point. The assumption M2 of (16) is valid also for this study. However, also curvatures around the contact patch must also be averaged by using weights. The same weight function given in Eq. 6 is also used. Only if this approach is used, approximation of the surfaces of the wheel and rail in the contact patch is obtained. Like Eq. 7 curvatures can be given as

\[
\bar{\kappa} = \frac{\int_{s_{\text{min}}}^{s_{\text{max}}} \kappa w(s, z_{\text{wheel}}, z_{\text{rail}}) ds}{\int_{s_{\text{min}}}^{s_{\text{max}}} w(s, z_{\text{wheel}}, z_{\text{rail}}) ds}.
\] (8)

where \( \bar{\kappa} \) is the curvature about the given axis. The wheel/rail contact points can be found in Fig. 4. The effect of quasi-elastic contact search on the contact points can be seen clearly in Fig. 5, as it smooths the characteristics.

**Fig. 4** Comparison of rigid and quasi-elastic contact search methods

**Fig. 5** Wheel/rail contact points in case of rigid and quasi-elastic search method
5. THEORY of HERTZ

The details of the theory of Hertz is given in (5). As in (6), (7), (8) approximation functions for the tabular data for the parameters consisting of solution of these integrals are widely used for faster calculations. In this work a numerical calculation method of these elliptical integrals will be given. The solution to find the shape (i.e. semi-axes of the contact area) requires to calculate complete elliptical integral of argument

\[ e = \sqrt{1 - \frac{b^2}{a^2}}, \]  

(9)

where \( b < a \) are semi-axis of the elliptical contact patch. As pressure acting on two bodies is same, equivalent elasticity modulus can be written as

\[ \frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}, \]  

(10)

where \( \nu_1, \nu_2 \) are Poisson ratios of materials. The pressure distribution is semi-ellipsoidal and from the known volume of ellipsoid, load \( P \) is given by

\[ P = \left( \frac{2}{3} \right) p_0 \pi ab. \]  

(11)

Calculation of contact areas requires knowledge of some geometric constants. These geometric constants calculated in terms of curvatures with respect to wheel/rail combination are given as

\[ A + B = \frac{1}{2} \left( \frac{1}{R_{1x}} + \frac{1}{R_{2x}} + \frac{1}{R_{1y}} + \frac{1}{R_{2y}} \right) \]  

(12)

\[ B - A = \frac{1}{2} \left( \frac{1}{R_{1x}} - \frac{1}{R_{1y}} \right)^2 + \left( \frac{1}{R_{2x}} - \frac{1}{R_{2y}} \right)^2 + 2 \left( \frac{1}{R_{1x}} - \frac{1}{R_{1y}} \right) \left( \frac{1}{R_{2x}} - \frac{1}{R_{2y}} \right) \cos^2 \psi \]  

(13)

In Eq. 13 the angle \( \psi \) is the angle between the axes of principal curvature of each surface. By using Eq. 12 and Eq. 13 \( A \) and \( B \) can be obtained easily. The illustration of curvatures can be found in Fig. 4.4 of (8). In order to find shape and size of elliptical contact, the following expression can be written (17)

\[ \frac{B}{A} = \left( \frac{O/A}{b} \right)^2 \frac{K(e) - K(e)}{K(e) - E(e)}, \]  

(14)

where \( e \) is the eccentricity of ellipse and \( K(e) \) and \( E(e) \) are first and second kind of elliptical integrals which are given in Appendix B of (8). Series expansion around the point \( e = 0 \) elliptical integrals for first and second kind can be computed as given in (18)

\[ K(e) = \frac{\pi}{2} \left[ 1 + \frac{1}{2} e^2 + \frac{1 \times 3}{2 \times 4} e^4 + \cdots + \left( \frac{(2n-1)!!}{2^n n!} \right)^2 e^{2n} + \cdots \right], \]  

(15)

\[ E(e) = \frac{\pi}{2} \left[ 1 - \frac{1}{2} e^2 - \frac{1 \times 3}{2 \times 4} e^4 - \cdots - \left( \frac{(2n-1)!!}{2^n n!} \right)^2 e^{2n} + \cdots \right]. \]  

(16)

In these equations \( n \) is the number of terms to approximate elliptical integrals. A more appropriate calculation model can be found in (19) as

\[ K(e) = \sum_{n=1}^{\infty} P_n \text{ where: } P_{n+1} = P_n \left( \frac{2n-1}{2n} \right)^2 e^2, \]  

(17)
\[ E(e) = \sum_{n=1}^{\infty} P_n \text{ where: } P_{n+1} = P_n \left( \frac{2n-1}{2n} \right)^2 e^{-\frac{2i-3}{2i-1}}. \] (18)

In both cases \( P_1 = \frac{\pi}{2} \). In each iteration \( e \) must be calculated. The initial value for calculation of \( e \) is taken as 0.5 in this study and the maximum iteration number is taken as \( 10^3 \). It is stated in (18) if \( e \) taken between the ranges \( 0 \leq e \leq 0.99 \) and taking just the four terms of the series for practical application, maximum percentage errors of Eq. 15 and Eq. 16 are 30 % and 6 % respectively. For the range \( 0 \leq e \leq 0.8 \) these errors reduce to 2 % and 0.4 %. In this study if the highest term is smaller than a threshold (e.g. \( 10^{-3} \)) iterations are terminated regardless of iteration number. Therefore more efficient and accurate calculation of elliptical integrals can be obtained. We need to calculate the value of eccentricity for given geometrical parameters \( A \) and \( B \) in each calculation by using the formula given below. This formula can be obtained by rearranging Eq. 9 and Eq. 14.

\[ e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{A}{B} \frac{E(e) - (1-e^2)K(e)}{K(e) - E(e)}} \] (19)

After each iteration the procedure is to control absolute difference between eccentricity values of each step. If this absolute difference is smaller or equal to a threshold (e.g. \( 10^{-3} \)) then it can be concluded that \( e \) is the eccentricity parameter for given geometrical parameters \( A \) and \( B \). Then it is not difficult to find the shape of the contact patch. By rearranging Eq. 9, 10, 11 and 14 semi-axis value \( a \) can be found as

\[ a = \sqrt{\frac{3}{2} \frac{K(e) - E(e)}{e^2 \pi A} \frac{P}{E(e) - E(\pi A)}}. \] (20)

Likewise semi-axis value \( b \) can be calculated. The results obtained for vertical load of 70 kN vertical load and lateral shift range \(-15 \leq y \leq 15 \) mm can be found in Fig. 6. Results found are compared with the results given by (13).

**Fig. 6 Semi axes values (quasi-elastic contact search)**

Literature

Summary

NONLINEAR WHEEL/RAIL CONTACT GEOMETRY CHARACTERISTICS & DETERMINATION OF HERTZIAN CONTACT

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In this study a procedure to obtain contact points and its characteristics is given. Widely used S1002 and UIC60E1 profile is used as an example to illustrate the proposed methodology. Firstly wheel/rail geometry is presented along with the calculation of contact points by considering roll angle of the wheelset. Afterwards a more realistic contact search method, namely quasi-elastic contact search, is presented. Lastly, the determination of the Hertzian contact patch is shown in details by giving numerical solution to elliptical integrals.