ON THE FIBONACCI NUMBERS AND F-POLYNOMIAL OF GRAPHS

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Abstract: *The theory of graph polynomials and their applications in several branches of science was developed by many authors. Some of these polynomials are related to the matching polynomial. The Fibonacci polynomial, shortly F-polynomial, is also an analogy the matching polynomial. We show the basic properties and methods of calculation of the F-polynomial. Furthermore, the Fibonacci number of a graph, including its determination for certain types of graphs, is mentioned.*

Keywords: *Simple Graph, Matching Polynomial, F-polynomial, Fibonacci Number*

1. Introduction

The characteristic polynomial and the matching polynomial have the basic position among the graph polynomials which are used not only in the graph theory but are also applied in various science branches including physics, chemistry or economics (see [6]). The study of spectral graph theory is concerned with the relationships with a graph and the topological properties of that graph. The spectrum of its adjacency matrix is called the spectrum of the graph. In some cases this spectrum determines the graph up to isomorphism. We can find more details in [1].

The graphs considered here are without loops and multiple edges. Let G = (V, E) be a simple graph with the vertex set $V = \{v_1, v_2, ..., v_p\}$ and the edge set E. A matching in G is a spanning subgraph of G whose components are isolated vertices and edges only. It means that a k-matching of G is a subset containing k edges of E such that no two edges have a vertex in common. Let p(G; k) = pk be the number of k – matchings of G with p0 = 1. The matching polynomial of a graph G can be defined by the relation (see e.g. [2])

$$M(G; w_1, w_2) = \sum_{k=0}^{[p/2]} p_k w_1^{p-2k} w_2^k .$$

Some authors use the special case when w1 = x and w2 = –1 as the acyclic polynomial. Certain interesting properties of the polynomial

$$M(G; x) = \sum_{k=0}^{[p/2]} (-1)^k p_k x^{p-2k}$$

were also given by Gutman and Harary in [3].

The matching polynomials of several classes of graphs are identical to typical orthogonal polynomials encountered in combinatorics. More concretely, the matching polynomial for a cycle corresponds to the Chebyshev polynomial of the first kind, for a path to the Chebyshev polynomial of the second kind, for a complete graph to the

Hermite polynomial and for a complete bipartite graph to the Laguerre polynomial (see e.g. [5]).

Sometimes it is more convenient to consider the positive matching polynomial

$$M^{+}(G;x)=\sum_{k=0}^{[p/2]} p_k x^k \; .$$

This is precisely the generating function for k – matching of a graph G.

It is a well-known fact that the total number of subsets of the set $\{1, 2,..., n\}$ such that no two elements are adjacent is F_{n+2}, where F_n is the n-th Fibonacci number given by the recurrence $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$. Therefore Prodinger and Tichy introduced in [7] the Fibonacci number f(G) of a graph G = (V; E) as the number of all subsets S of V such that no two vertices in S are adjacent in G. Several authors studied the Fibonacci numbers of various types of graphs (also [9]).

Now, denote s(G; k) = sk the number of selections of k independent vertices in G and define $s_0 = 1$. Then the F-polynomial (Fibonacci polynomial) of a graph G is defined in [4] by the relation

$$F(G;x)=\sum_{k=0}^{t} s_k x^k \; ,$$

where l is the cardinality of the largest independent vertex set. It is easy to realize that the equality F(G; 1) = f(G) is true for every graph G.

The definitions of the polynomials M + (G; x) and F(G; x) have the following immediate consequence.

Theorem 1. *For every graph G the relation*

$$M^{+}(G; x) = F(L(G); x)$$

holds, where L(G) denotes as usual the line graph of G, which has a vertex of L(G) associated with each edge of G and an edge of L(G) exists if and only if the two edges of G share a common vertex.

Then, it is easy to see that F(Cp; x) = M +(Cp; x) and F(Pp; x) = M +(Pp+1; x), where Cp and Pp denote the cycle and the path with p vertices. Thus the F-polynomial is a generalization of the positive matching polynomial of a graph.

2. Basic properties of the F-polynomial

Questions about the number of independent (not adjacent) vertices of a graph are among the classic problems of the graph theory. Simple combinatorial arguments yield the next statements (see e.g. [3]).

Theorem 2. *If G is a graph with p vertices, q edges, t triangles (cycles of the length 3) and if it has the degree sequence (d_1, d_2, …, d_p), then*

$$s(G;1)=p,\ \ s(G;2)=\binom{p}{2}-q;\ \ s(G;3)=\binom{p}{3}-q(p-2)+\ \sum_{i=1}^{p}\binom{d_i}{2}-t\,.$$

Theroem 3. *For two disjoint graphs G_1 and G_2 the relation*

$$F(G_1\cup G_2;x)=F(G_1;x)\cdot F(G_2;x)$$

holds.

The following statement is very helpful if we create an algorithm for calculation of the F-polynomial.

Theorem 4. *If v is an arbitrary vertex of a graph G. Then*

$$F(G;x)=F(G-v;x)+xF(G-(v);x)\,,$$

where $G - v$ is the subgraph of G obtained by deletion of the vertex v and $G - (v)$ is the subgraph of G obtained by deletion of the vertex v and all the vertices adjacent to v.

Proof. We construct a decomposition of the set of all selections of k independent vertices in G into two disjoint subsets with respect to the fact of whether the selection contains the given vertex v or not. Then we obtain the relation

$$s(G;k)=s(G-v;k)+s(G-(v);k-1).$$

Now, the statement is obvious using the definition of the F-polynomial.

Theorem 5 ([3], Proposition 8). *Let $v_1, \ldots, v_p$ be the vertices of a graph G. Then*

$$\frac{d}{dx}F(G;x)=\sum_{i=1}^{p}F(G-(v_i);x).$$

3. Calculation of the F-polynomial of simple graphs

We can calculate the F-polynomial using its definition only for the graphs with a small number of vertices. Furthermore, this method is also suitable for some special classes of graphs.

Theorem 6. *The F-polynomials for the complete graph K_p with p vertices, its complement $\overline{K_p}$ and the complete bipartite graph K_{p_1,p_2} are the following*

$$F(K_p;x)=1+px,$$

$$F(\overline{K_p};x)=(1+x)^p,$$

$$F(K_{p_1,\ p_2};x)=(1+x)^{p_1}+(1+x)^{p_2}-1.$$

Proof. All vertices of the complete graph are mutually adjacent and therefore $s(K_p;1)=p,\ s(K_p;k)=0$ for any $k>1$. The graph $\overline{K_p}$ has no edges and no adjacent vertices. Therefore $s(\overline{K_p};k)=\binom{p}{k}$ as the number of the subsets of V with k elements. The binomial theorem gives the equality $\sum_{k=0}^{p}\binom{p}{k}x^k=(1+x)^p$. The vertex set of the complete bipartite graph is divided into two subsets with p_1, p_2 vertices such that edges only exist between vertices of the different subsets. It is easy to see that $s(K_{p_1, p_2}; k) = \binom{p_1}{k} + \binom{p_2}{k}$ using for the binomial coeficients the convention $\binom{p}{k} = 0$ if $p < k$. Then $F(K_{p_1, p_2}; x) = 1 + \sum_{k=1}^{p_1} \binom{p_1}{k} x^k + \sum_{k=1}^{p_2} \binom{p_2}{k} x^k$ and the proof is over.

Corollary 7. *The F-polynomial for a star S_p with p + 1 vertices has the form*

$$F(S_p; x) = \sum_{k=0}^{p} \binom{p}{k} x^k + x = (1+x)^p + x.$$

Proof. The star S_p is a tree having one vertex of the degree p and the other vertices have the degree equal to one. It is obvious that $s(S_p; k) = \binom{p}{k}$ for $k = 0$ and $2 \le k \le p$, further $s(S_p;1) = \binom{p}{1} + 1$. This result agrees with the before given form of the *F*-polynomial for the complete bipartite graphs as $S_p = K_{p,1}$.

However, a similar computation for other classes of graphs can be more complicated. In that case, it is possible to use Theorem 4. We will apply it to the path Pp and the cycle Cp with p vertices. We choose as the vertex v the vertex of the degree one for the path Pp and we choose its arbitrary vertex v in the case of the cycle Cp.

Then the following recurrences hold

$F(P_p; x) = F(P_{p-1}; x) + xF(P_{p-2}; x),$ where $p \ge 2$,

$F(C_p; x) = F(P_{p-1}; x) + xF(P_{p-3}; x),$ where $p \ge 3$,

with $F(P_0; x) = 1, F(P_1; x) = x + 1.$

By rewriting of these relations to the commands of the system Mathematica we have obtained the F-polynomials for paths Pp and cycles Cp , where p = 2, …, 10 (see Table 1, Table 2).

Table 1: F-polynomial of paths P_p for small values of p

p	$F(P_p; x)$
2	$1 + 2x$
3	$1 + 3x + x^2$
4	$1 + 4x + 3x^2$
5	$1 + 5x + 6x^2 + x^3$
6	$1 + 6x + 10x^2 + 4x^3$
7	$1 + 7x + 15x^2 + 10x^3 + x^4$
8	$1 + 8x + 21x^2 + 20x^3 + 5x^4$
9	$1 + 9x + 28x^2 + 35x^3 + 15x^4 + x^5$
10	$1 + 10x + 36x^2 + 56x^2 + 35x^4 + 6x^5$

Table 2: F-polynomial of cycles Cp for small values of p

p	$F(C_p; x)$
3	$1 + 3x$
4	$1 + 4x + 2x^2$
5	$1 + 5x + 5x^2$
6	$1 + 6x + 9x^2 + 2x^3$
7	$1 + 7x + 14x^2 + 7x^3$
8	$1 + 8x + 20x^2 + 16x^3 + 2x^4$
9	$1 + 9x + 27x^2 + 30x^3 + 9x^4$
10	$1 + 10x + 35x^2 + 50x^3 + 25x^4 + 2x^5$

Theorem 4 makes it also possible to create a general algorithm for calculation of the F-polynomial of an arbitrary simple graph G by the system Mathematica. A graph G with the vertices $v_1, v_2, ..., v_p$ can be uniquely characterized by its adjacency matrix $A = (a_{ij}), i, j = 1,... p$. It is advantageous to choose as the vertex v in Theorem 4 the vertex vk having the highest degree. Then the adjacency matrix of the graph G – vk is obtained from the matrix A by deletion of the k-th row and the k-th column. Similarly the adjacency matrix of the graph G – (vk) is obtained from the matrix A by deletion of the k-th row, the k-th column and all the rows and columns which correspond to the vertices adjacent to vk. Further Theorem 4 is used repeatedly on the graphs G – vk , G – (vk) and the graphs obtained during the procedure. The algorithm finishes when the graphs with the smallest number of vertices are acquired. The “empty” adjacency matrix (having no row and column) corresponds to the “empty” graph with the F-polynomial equal to one. More details about this program in [10].

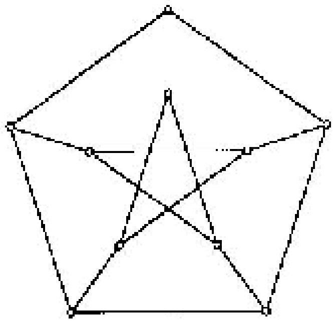

Fig. 1

The so-called the Petersen graph G1 is drawn in Fig.1 and the polycyclic graph G2 with ten vertices in Fig. 2. Their F-polynomials calculated by the given program are

F(G1; x) = 1 + 10x + 30x2 + 30x3 + 5x4, F(G2; x) = 1 +10x + 25x2 + 6x3.

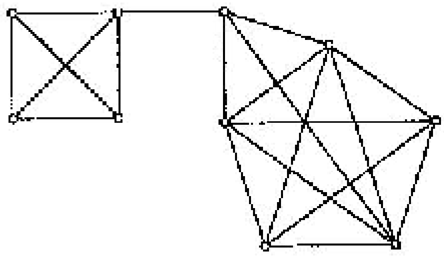

Fig. 2

4. Concluding remark

Using the equality F(G; 1) = f(G) makes it possible to calculate the Fibonacci number of a graph if the F-polynomial is given. These values are also interesting for cycles. We mentioned in the first section that $f(P_p) = F_{p+2}$. From the recurrence $F(C_p; x) = F(P_{p-1}; x) + xF(P_{p-3}; x),$ $p \geq 3$, we have $f(C_p) = f(P_{p-1}) + f(P_{p-3}) = F_{p+1} + F_{p-1} = L_p,$ where L_p is the p-th Lucas number as the term of the integer sequence $\{1, 3, 4, 7, 11, \ldots\}$.

A very useful task is also to find the zeros of some graph polynomials. It is known that the zeros of the matching polynomial M(G; x) are real and therefore the zeros of M + (G; x) must be negative real numbers. But the zeros of the F-polynomial can be complex numbers.. A question arises. Is it possible to find some important classes of graphs whose F-polynomial has only real zeros?

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