THE SIERPIŃSKI TRIANGLE AND ITS COORDINATE FUNCTIONS

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Abstract: *The famous fractal set called the Sierpiński triangle was introduced as a plane curve every point of which is the point of ramification. Since it satisfies the Jordan definition of a curve, it can be represented by two continuous coordinate functions of a parameter. The coordinate functions are constructed by iterations of a system of linear transformations in the complex plane.*

Keywords: *Sierpiński Triangle, Jordan Curve, Fractals*

1. Introduction

Many examples of sets with strange and counter-intuitive properties appeared during the development of point set theory in the first decades of the 20th century. The general definition of a curve had been missing until 1920's. There were two widely accepted definitions of a curve: the Jordan definition which describes a curve parametrically and Zoretti's definition of so called Cantorian line, i.e. a continuum which is nowhere dense in the plane. However, both definitions allowed sets that are far from intuitive understanding of the concept of a curve. The classes of Jordan curves and Cantorian lines are not identical; there are Cantorian lines that do not satisfy Jordan's definition and vice versa. Even the sets that can be called lines according to both definitions may have very peculiar properties.

In the note [2] presented to the Academy of Sciences in Paris in 1915, Polish mathematician Wacław Sierpiński described a plane set which satisfies both definitions whose every point is the point of ramification. Thus begins the history of the celebrated set which is now known as the Sierpiński triangle or the Sierpinski gasket. Its complex structure contrasts with apparent simplicity of its construction. It is constructed from an equilateral triangle T by a sequence of deletion operations. The initial triangle is divided into four smaller equilateral triangles and the inner points of the middle triangle U are removed (Fig. 1). The set F1 thus consists of three triangles T0, T1, T2. The same operation is repeated with each of them. The set F2 is the union of nine equilateral triangles T00,..., T22. The process continues ad infinitum. The Sierpiński triangle is the set F which consists of all points that all sets Fk, $k \in N$ have in common.

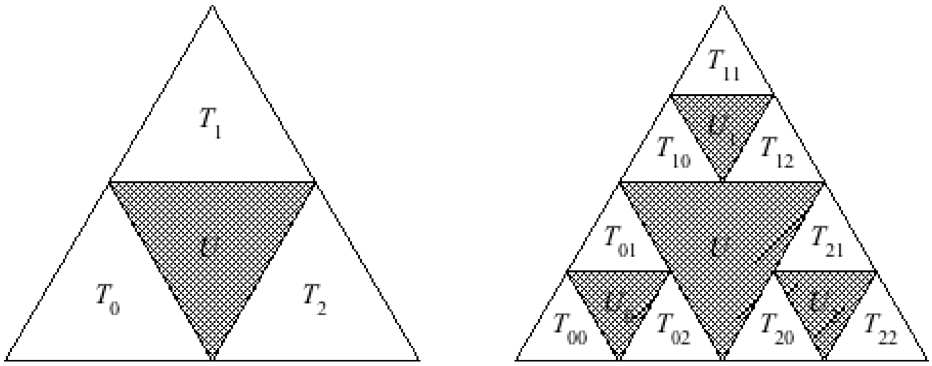

Fig. 1. Construction of the Sierpiński triangle.

2. Approximating polygons

The set F is nowhere dense in the plane and hence it is a Cantorian line. It can be shown that F is also the Jordan curve. In the extended Polish version [3] of the above mentioned note Sierpiński shows how to represent F in terms of approximating polygons. If we place the initial triangle T into the complex plane so that its left corner coincides with the origin and the real axis points in the direction of its base of unit length, then the polygonal lines are constructed as follows. Let L1 be a polygonal line passing through the points $z_1^{(0)} = 0$, $z_1^{(1)} = \frac{1}{4} + \frac{\sqrt{3}}{4}i$, $z_1^{(2)} = \frac{3}{4} + \frac{\sqrt{3}}{4}i$, $z_1^{(3)} = 1$ (Fig. 2). It is an initial line which Sagan [1, p. 23] calls leitmotiv. The line L2 is obtained from L1 by replacing its sides with three copies of itself placed in the triangles T01, T02 and T03 (Fig. 1) so that the resulting line is connected. Sierpiński does not use complex representations of points in the plane, but his construction is essentially the same.

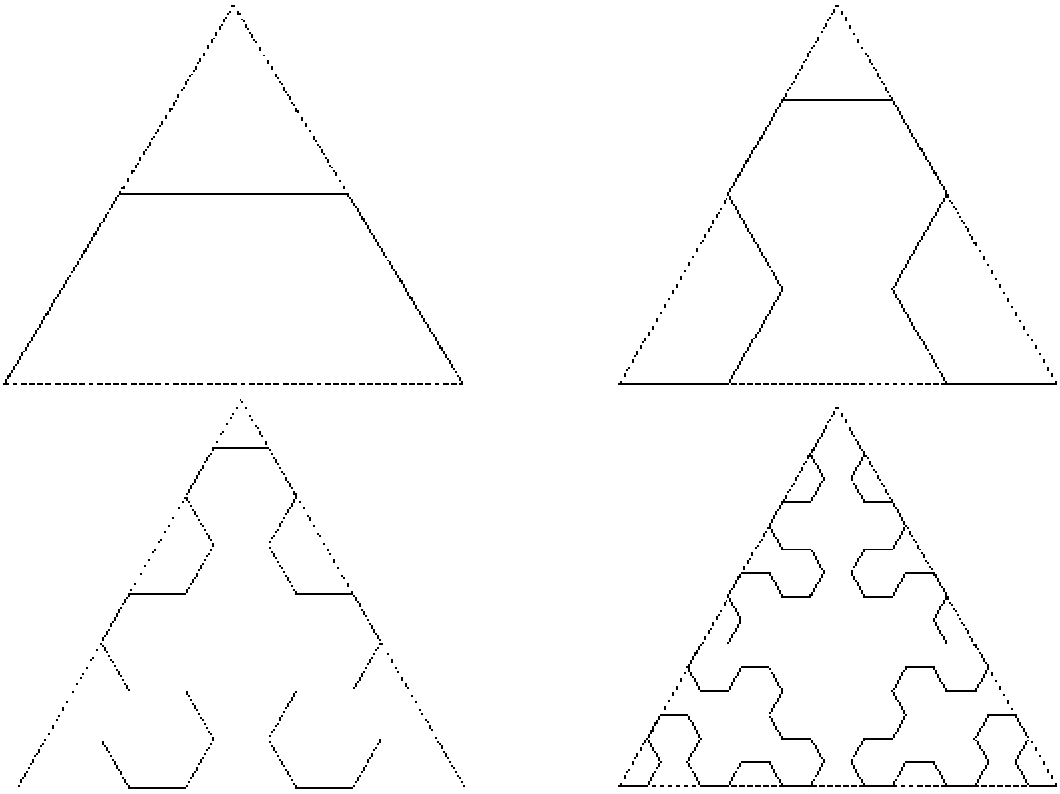

Fig. 2. Approximating polygons for the Sierpiński triangle.

The polygonal line Ln is obtained recursively from the leitmotiv and passes through 3^n+1 points $z_n^{(0)}$, $z_n^{(1)}$,..., $z_n^{(3^n)}$. It can be expressed by equations

$$\left.\begin{aligned} x &= j_n(t) \\ y &= y_n(t) \end{aligned}\right\} \quad t \in [0,1] \tag{1}$$

so that the values

$$t_n = 0, \frac{1}{3^n}, \frac{2}{3^n}, \mathbf{K}, \frac{3^n-1}{3^n}, 1 \tag{2}$$

correspond with $z_n^{(0)}$, $z_n^{(1)}$,..., $z_n^{(3^n)}$ and the functions $x = j_n(t)$, $y = y_n(t)$ are linear in every interval $\left(\frac{k}{3^n}, \frac{k+1}{3^n}\right)$, $k = 0, \mathbf{K}, 3^n-1$. Sierpiński demonstrates that the sequence of polygons converges uniformly and the limiting set is the set F, which is therefore a continuous image of the unit interval, i.e. the Jordan curve.

3. The Coordinate Functions

The set F consists of three small copies of itself and thus it can be taken as an invariant set of an iterated function system (IFS) composed of three contraction maps

$$S_0(z)=\frac{1}{2}w\bar{z},\quad S_1(z)=\frac{1}{2}(z+w),\qquad S_2(z)=\frac{1}{2}(\bar{w}\bar{z}+w+1) \tag{3}$$

where $w=e^{i\frac{\pi}{3}}$. The above construction can be expressed by means of transformations S0, S1, S2. If we start again with the initial line L1, then $L_2=S_0(L_1)\cup S_1(L_1)\cup S_2(L_1)$ and generally for every positive integer n we have

$$L_{n+1}=S_0(L_n)\cup S_1(L_n)\cup S_2(L_n). \tag{4}$$

To obtain the coordinate functions (1) for the polygonal line Ln we express the values (2) of the parameter t as

$$t_n=\frac{j_1}{3}+\frac{j_2}{3^2}+\ldots+\frac{j_n}{3^n},$$

where the numerators take the values 0,1 or 2, or, which is the same, as the number $(0,j_1 j_2\ldots j_n)_3$ in the triadic system. Every value of $t_n^{(k)}$, $k=0,\ldots,3^n-1$ corresponds with a unique sequence of transformations (3) which maps the point 0 to $z_n^{(k)}$. For example, the point $t_2^{(5)}=(0{,}12)_3=\frac{5}{9}$ is mapped to $z_2^{(5)}=S_1\circ S_2(0)$, etc. Generally,

$$z_n^{(k)}=S_{j_1}\circ S_{j_2}\circ\ldots\circ S_{j_n}(0) \tag{5}$$

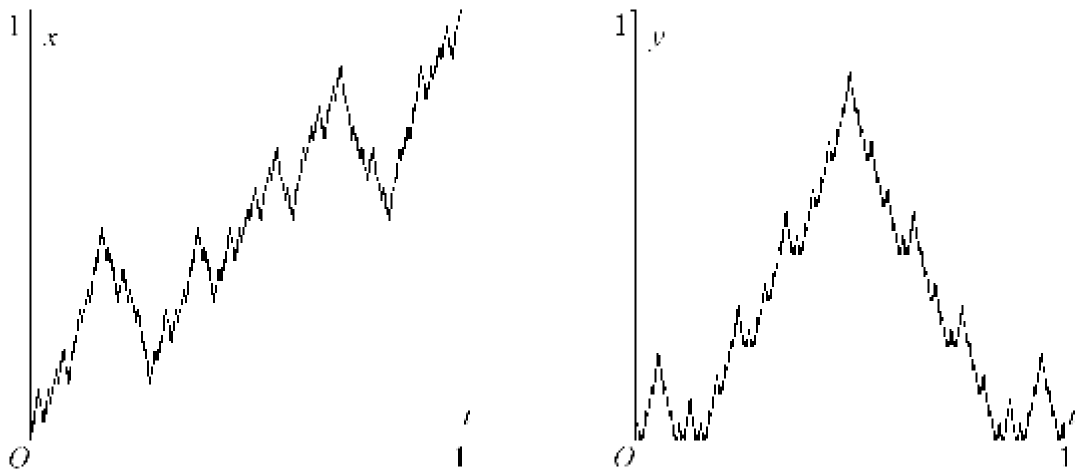

Fig. 3. Approximations of graphs of coordinate functions.

The polygon Ln thus can be taken as the image of the unit interval

$$z=f_n(t),\quad t\in[0,1], \tag{6}$$

where the vertices $z_n^{(k)}$ are given by (5) and f(t) is linear in every interval $\left(\frac{k}{3^n}, \frac{k+1}{3^n}\right)$, $k = 0, \mathbf{K}, 3^n - 1$. The real and imaginary parts of (6) are the functions $x = j_n(t)$, $y = y_n(t)$, respectively. Their graphs for $n = 5$ are depicted on Fig. 3. Both graphs give a good visualisation of the shape of coordinate functions of the Sierpiński triangle $x = j(t) = \lim_{n\to\infty} j_n(t)$ and $y = y(t) = \lim_{n\to\infty} y_n(t)$ for $t \in [0,1]$. The graphs show some form of self-similarity of φ(t) and ψ(t) (although not strict) and fine structure and can be therefore taken as examples of fractal curves.

4. Conclusion

Coordinate functions of Peano, Osgood and other special types of curves were studied long before the emergence of the fractal theory (see for example [1, p. 51 ff.]). However, their significance grew considerably as they served as examples of self-affine and other fractal curves. Functions whose graphs are fractal curves are useful to study various phenomena, including, for example, behaviour of stock markets. It has been observed that the graphs of price variations may contain patterns that are scale-independent and can be thus regarded as statistically self-similar fractal curves. Models based on the fractal theory have been used to describe the behaviour of financial markets and to explain the existence of extreme fluctuations of prices. The Sierpiński triangle is one of the best known fractals and found its way even into areas outside mathematics.

References:

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[3] SIERPIŃSKI, W. *O krzywej, której każdy punkt jest punktem rozgałęzienia*. Prace matematyczno-fizyczne 27 (1916), 77–86.

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