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ON ONE TWO-POINT BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF EVOLUTIONAL DIFFERENTIAL EQUATIONS

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In this paper there are established effective criteria for the existence and uniqueness of the solution of one two-point boundary value problem for the system of differential equations with delays

$$\frac{dx(t)}{dt} = f(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t)))$$

$$x(0) = (1-\mu)x(\ell) + \mu c, \quad x(t) = \varphi(t) \text{ for } t < 0,$$

where for a natural number n, an integer m, $\mu \in [0, 1]$, *and the interval* $I = [0, \ell] \subset R$, *the function* $f : I \times R^{n(1+m)} \to R^n$ *is a vector-valued function satisfying the local Carathčodory conditions,* $\tau_j : I \to R$ $(j = 1, \dots, m)$ *are measurable functions such that* $\tau_j(t) \leq t$ $(j = 1, \dots, m)$ *for almost all* $t \in I$,

$c \in R^n$, and $\varphi :]-\infty, 0[\to R^n$ is a continuous and bounded vector-valued function.

Formulation of Problem

Let n be a natural number, m be a nonnegative integer, $I = [0, \ell]$ be a closed interval of real numbers, and $\mu \in [0, 1]$. Assume that the vector-valued function $f : I \times R^{n(1+m)} \to R^n$ satisfies the local Carathčodory, i.e.,

1) the function $f(\cdot, x_0, x_1, ..., x_m) \in L(I; R^n)$ for any $x_0, x_1, ..., x_m \in R^n$,

2) the function $f(t, \cdot) : R^{n(1+m)} \to R^n$ is continuous for almost all $t \in I$

3) $\sup\{\|f(t, x_0, x_1, ..., x_m)\| : x_0, x_1, ..., x_m \in R^n, \sum_{j=0}^{m} \|x_j\| \le \rho\} \in L(I; R)$

Consider the boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t), x(\tau_1(t)), ..., x(\tau_m(t))) \tag{1.1}$$

$$x(0) = (1 - \mu)x(\ell) + \mu c \tag{1.2}$$

$$x(t) = \varphi(t) \quad \text{for } t \in]-\infty, 0[. \tag{1.2'}$$

Under a solution to the system (1.1) we understand an absolutely continuous vector-valued function $x : I \to R^n$, which satisfies Eq. (1.1) almost everywhere on I. Under the solution to the problem (1.1), (1.2), (1.2′) we understand the solution to the system (1.1) satisfying the conditions (1.2), (1.2′).

Note 1.1. If $m = 0$, then the system (1.1) is the system of ordinary differential equations (without any deviated argument).

Note 1.2. If $m \neq 0$ and $\tau_j(t) \in I$ for almost all $t \in I$ and every $j = 1, ..., m$, then the condition (1.2′) becomes unimportant and the boundary value problem (1.1), (1.2), (1.2′) is the boundary value problem (1.1), (1.2).

Note 1.3. Note also that every continuous vector-valued function (i.e., continuous in all variables) satisfies the Carathčodory conditions.

Special cases of the problem considered are often studied in the literature that deals with the applications of differential equations, e.g., the mathematical model of epidemic (D. Bernoulli – 1760, Kermack, MacKendrick – 1927), the kinetics of the fermentation (Okamoto, Nagashi – 1984), of the immunological processes (Marchuk – 1975), or of the chemical reactor (Finkelstein – 1971). In most cases there are present either the systems of ordinary differential equations (1.1) with the continuous right-hand side, or the systems with very particular delayed arguments – especially with constant delays. The present paper supplies and extends the domain of the solvable problems in this respect.

General questions of solvability of the boundary value problems of functional differential equations are studied, e.g., in monograph [1], the effective criteria for the solvability can be found in Refs [2,4,6] (as for the differential equations with deviating argument see also Ref. [5]) and references cited therein.

The following notation is used throughout.

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$, $I = [0, \ell]$.

R^n is the space of n-dimensional real column vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in R$ $(i = 1, \ldots, n)$ and the norm

$$\|x\| = \sum_{i=1}^{n} |x_i| \ .$$

$R_+^n = \{x = (x_i)_{i=1}^n \in R^n : x_i \geq 0,\ i = 1, \ldots, n\}$.

If $x, y \in R^n$, then $x \leq y \Leftrightarrow y - x \in R_+^n$, $\operatorname{sgn} x = (\operatorname{sgn} x_i)_{i=1}^n$, $|x| = (|x_i|)_{i=1}^n$.

$R^{n\times n}$ is the space of real $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with the components x_{ik} $(i, k = 1, \ldots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^{n} |x_{ik}| \ .$$

$R_+^{n\times n} = \{X = (x_{ik})_{i,k=1}^n \in R^{n\times n} : x_{ik} \geq 0,\ i, k = 1, \ldots, n\}$.

If $X, Y \in R^{n\times n}$, then $X \leq Y \leftrightarrow Y - X \in R_+^{n\times n}$, $|X| = (|x_{ik}|)_{i,k=1}^n$.

If $x = (x_i)_{i=1}^n$, then $\operatorname{diag}(x)$ is the diagonal $n \times n$-matrix with the diagonal components $x_1, \ldots, x_n$.

$C(I; R^n)$ is the space of continuous vector-valued functions $x : I \to R^n$ with the norm

$$\|x\|_C = \max\{\|x(t)\| : t \in I\} \ .$$

If $x \in C(I; R^n)$, then $|x|_C = (\|x_i\|_C)_{i=1}^n$.

$L^q(I; R^n)$, where $1 \le q < +\infty$, is the space of vector-valued functions $x : I \to R^n$, whose components can be integrated with the q-th power, and with the norm

$$\|x\|_{L^q} = \left(\int_0^\ell \|x(t)\|^q dt \right)^{1/q} .$$

$L^q(I; R_+^n) = \{x \in L^q(I; R^n) : x(t) \ge 0 \text{ for almost all } t \in I\}$.
$L^q(I; R^{n\times n})$, where $1 \le q < +\infty$, is the space of matrix-valued functions $x : I \to R^{n\times n}$, whose components can be integrated with the q-th power, and with the norm

$$\|X\|_{L^q} = \left(\int_0^\ell \|X(t)\|^q dt \right)^{1/q} .$$

$L^q(I; R_+^{n\times n}) = \{X \in L^q(I; R^{n\times n}) : X(t) \ge 0 \text{ for almost all } t \in I\}$.
If $x \in L^q(I; R^n)$, $X \in L^q(I; R^{n\times n})$, then $|x|_{L^q} = (\|x_i\|_{L^q})_{i=1}^n$, $|X|_{L^q} = (\|x_{ik}\|_{L^q})_{i,k=1}^n$.
χ_I is the characteristic function of the interval I, i.e.,

$$\chi_I(t) = \begin{cases} 1 & \text{for } t \in I \\ 0 & \text{for } t \notin I \end{cases}$$

and

$$\tau^0(t) = \begin{cases} 0 & \text{for } \tau(t) < 0 \\ \tau(t) & \text{for } 0 \le \tau(t) \le \ell \end{cases} .$$

Denote by

$$f_0(t, x, x_1, \ldots, x_m) = f(t, x, \chi_I(\tau_1(t))x_1 + (1 - \chi_I(\tau_1(t)))\varphi(\tau_1(t)), \ldots$$
$$\ldots, \chi_I(\tau_m(t))x_m + (1 - \chi_I(\tau_m(t)))\varphi(\tau_m(t))) ,$$

and together with the system (1.1) consider the system

$$\frac{dx(t)}{dt} = f_0(t, x(t), x(\tau_1^0(t)), \ldots, x(\tau_m^0(t))) \tag{1.3}$$

Then every solution x to the problem (1.1), (1.2), (1.2′) is also a solution to the problem (1.3), (1.2), and *vice versa*, i.e., the problems (1.1), (1.2), (1.2′) and (1.3), (1.2) are equivalent.

Main Results

Theorem 2.1. *Let there exist $\rho > 0$ and the matrix-valued function $P \in L(I; R^{n \times n})$ such that the boundary value problem*

$$\frac{dx(t)}{dt} = P(t)x(t) \tag{2.1}$$

$$x(0) = (1-\mu)x(\theta) \tag{2.2}$$

has only the trivial solution and every solution x to the boundary value problem

$$\frac{dx(t)}{dt} = P(t)x(t) + \lambda\left[f_0(t, x(t), x(\tau_1^0(t)), \ldots, x(\tau_m^0(t))) - P(t)x(t)\right] \tag{2.3}$$

$$x(0) = (1-\mu)x(\theta) + \lambda\mu c, \tag{2.4}$$

where $\lambda \in [0, 1]$ admits the estimate

$$\|x\|_C \le \rho. \tag{2.5}$$

Then the boundary value problem (1.1), (1.2), (1.2′) has at least one solution.

Now let $P(t) = \mathrm{diag}(-p(t))$, where the vector-valued function $p = (p_1, \ldots, p_n) \in L(I; R_+^n)$. The specification of the conditions deals with the function f and yields the following effective criteria of the existence and uniqueness of the solution to the problem considered.

Theorem 2.2. *Let for almost all $t \in I$ and every $x, x_1, \ldots, x_m \in R^n$ the inequality*

$$diag(sgn x) f(t, x, x_1, \dots, x_m) \le$$
$$\le diag(-p(t))|x| + P(t)|x| + \sum_{j=1}^{m} P_j(t)|x_j| + \omega(t) \tag{2.6}$$

be fulfilled, where $p, \omega \in L(I; R_+^n)$, $P, P_j \in L^q(I; R_+^{n\times n})$, $(j = 1, \dots, m)$, $1 \le q < +\infty$. *Let, moreover, the spectral radius of the matrix*

$$S = diag((\alpha_i)_{i=1}^n)\left[|P|_{L^q} + \sum_{j=1}^{m} |\chi_I(\tau_j)P_j|_{L^q}\right] \tag{2.7}$$

be strictly less than one, where

$$\alpha_i = \frac{(1-\mu)\ell^{1/p}}{1-(1-\mu)\exp(-\|p_i\|_L)} + \left(\frac{2\ell}{\pi}\right)^{2/p} \quad (i = 1, \dots, n),$$
$$(1-\mu)\exp(-\|p_i\|_L) < 1 \quad (i = 1, \dots, n) \tag{2.8}$$

and $\frac{2}{p} + \frac{1}{q} = 1$. *Then the boundary value problem (1.1), (1.2), (1.2') has at least one solution.*

<u>*Theorem 2.3.*</u> *Let for almost all* $t \in I$ *and every* $x, y, x_1, y_1, \dots, x_m, y_m \in R^n$ *the inequality*

$$diag(sgn(x-y))[f(t, x, x_1, \dots, x_m) - f(t, y, y_1, \dots, y_m)] \le$$
$$\le diag(-p(t))|x-y| + P(t)|x-y| + \sum_{j=1}^{m} P_j(t)|x_j - y_j| \tag{2.9}$$

be fulfilled, where $p \in L(I; R_+^n)$, $P, P_j \in L^q(I; R_+^{n\times n})$, $(j = 1, \dots, m)$, $1 \le q < +\infty$, *and the spectral radius of the matrix S defined by (2.7) be strictly less than one. Then the boundary value problem (1.1), (1.2), (1.2') has a unique solution.*

If we choose $\mu = 0$, and/or $\mu = 1$ and specify the vector-valued function p and the matrix-valued functions P, P_j ($j = 1, \dots, m$), from Theorems 2.2 and 2.3 we obtain the following corollaries on the existence and uniqueness of the solution to the periodic and/or initial problem.

Corollary 2.1. *Let for almost all $t \in I$ and every $x, x_1, ..., x_m \in R^n$ [and $y, y_1, ..., y_m \in R^n$] the inequality (2.6) [(2.9)] be fulfilled, where $p, \omega \in L(I; R_+^n)$, $P, P_j \in L^q(I; R_+^{n\times n})$, $(j = 1, ..., m)$, $1 \le q < +\infty$, $\|p_i\|_L > 0$ $(i = 1, ..., n)$ and the spectral radius of the matrix*

$$S = diag\left(\left(\ell^{1/p}[1 - \exp(-\|p_i\|_L)]^{-1} + \left(\frac{2\ell}{\pi}\right)^{2/p}\right)_{i=1}^{n}\right) \times$$

$$\times \left[|P|_{L^q} + \sum_{j=1}^{m} |\chi_I(\tau_j) P_j|_{L^q}\right]$$

be strictly less than one, and $\frac{2}{p} + \frac{1}{q} = 1$. Then the system (1.1) has, on the interval I, at least one [a unique] solution satisfying the periodic boundary condition

$$x(0) = x(\ell) .$$

Corollary 2.2. *Let for almost all $t \in I$ and every $x, x_1, ..., x_m \in R^n$ [and $y, y_1, ..., y_m \in R^n$] the inequality (2.6) [(2.9)] be fulfilled, where $p, \omega \in L(I; R_+^n)$, $P, P_j \in L^q(I; R_+^{n\times n})$, $(j = 1, ..., m)$, $1 \le q < +\infty$, $\|p_i\|_L > 0$ $(i = 1, ..., n)$ and the spectral radius of the matrix*

$$S = |P|_{L^q} + \sum_{j=1}^{m} |\chi_I(\tau_j) P_j|_{L^q}$$

be strictly less than $\left(\frac{\pi}{2\ell}\right)^{2/p}$, and $\frac{2}{p} + \frac{1}{q} = 1$. The system (1.1) has, on the interval I, at least one [a unique] solution satisfying the initial condition

$$x(0) = c$$

for any $c \in R^n$.

In the case where $\mu \in (0, 1]$ we can put $p \equiv 0$ on I, and for the constant matrix-valued functions P, P_j $(j = 1, ..., m)$ we obtain

Corollary 2.3. *Let for almost all $t \in I$ and every $x, x_1, ..., x_m \in R^n$ [and $y, y_1, ..., y_m \in R^n$] the inequality*

$$diag(sgn x) f(t, x, x_1, \ldots, x_m) \le$$

$$\le P|x| + \sum_{j=1}^{m} P_j |x_j| + \omega(t)$$

$$\left[diag(sgn(x-y)) [f(t, x, x_1, \ldots, x_m) - f(t, y, y_1, \ldots, y_m)] \le \right.$$

$$\left. \le P|x-y| + \sum_{j=1}^{m} P_j |x_j - y_j| \right]$$

be fulfilled, where $\omega \in L(I; R_+^n)$, $P, P_j \in R_+^{n \times n}$, $(j = 1, \ldots, m)$ *and the spectral radius of the matrix*

$$S = P + \sum_{j=1}^{m} \|\chi_I(\tau_j)\|_L P_j$$

be strictly less than $\left[\frac{1-\mu}{\mu} \ell^{3/4} + \left(\frac{2}{\pi} \right)^{1/2} \ell \right]^{-1}$. *Then the system (1.1) has, on the interval I, at least one [a unique] solution satisfying the boundary condition*

$$x(0) = (1-\mu) x(\ell) + \mu c$$

for any $c \in R^n$.

For the case when (1.1), (1.2) is the boundary value problem without any deviation from the above-mentioned results we obtain criteria which are known from the classical literature.

<u>Corollary 2.4.</u> *Let for almost all* $t \in I$ *and every* $x \in R^n$ *[and* $y \in R^n$*] the inequality*

$$diag(sgn x) f(t, x) \le diag(-p(t))|x| + P(t)|x| + \omega(t)$$

$$[diag(sgn(x-y)) [f(t, x) - f(t, y)] \le diag(-p(t))|x-y| + P(t)|x-y|]$$

be fulfilled, where $p, \omega \in L(I; R_+^n)$, $P \in L^q(I; R_+^{n \times n})$, $1 \le q < +\infty$, *and the spectral radius of the matrix* $/P/_{L^q}$ *be strictly less than*

$$\left[\frac{(1-\mu)\ell^{1/p}}{1-(1-\mu)\alpha} + \left(\frac{2\ell}{\pi} \right)^{2/p} \right]^{-1}$$

where $\alpha = \max\{\exp(-\|p_i\|_I) : i = 1, ..., n\}$ *and* $\frac{2}{p} + \frac{1}{q} = 1$. *Then the problem*

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad x(0) = (1 - \mu)x(\ell) + \mu c$$

has, on the interval I, *at least one [a unique] solution for any* $c \in R^n$.

Proofs

Proof of Theorem 2.1. Let $f : C(I; R^n) \to L(I; R^n)$ *and* $h : C(I; R^n) \to R^n$ be, in general, nonlinear continuous operators such that for every $\rho \in R_+$

$$\sup\{\|f(x)(\cdot)\| : x \in C(I; R^n), \|x\|_C \le \rho\} \in L(I; R_+)$$

and

$$\sup\{\|h(x)\| : x \in C(I; R^n), \|x\|_C \le \rho\} < +\infty.$$

According to Corollary 2 [5], if there exists $\rho > 0$, a linear strongly bounded operator $p_0 : C(I; R^n) \to L(I; R^n)$, and a linear bounded operator $\ell_0 : C(I; R^n) \to R^n$ such that the boundary value problem

$$\frac{dx(t)}{dt} = p_0(x)(t), \quad \ell_0(x) = 0 \tag{3.1}$$

has only the trivial solution, and every solution x to the boundary value problem

$$\frac{dx(t)}{dt} = p_0(x)(t) + \lambda[f(x)(t) - p_0(x)(t)], \quad \ell_0(x) = \lambda[\ell_0(x) - h(x)], \tag{3.2}$$

where $\lambda \in [0, 1]$ admits the estimate (2.5), then the boundary value problem

$$\frac{dx(t)}{dt} = f(x)(t), \quad h(x) = 0 \tag{3.3}$$

has at least one solution.

Put

$$f(x)(t) = f_0(t, x(t), x(\tau_1^0(t)), \dots, x(\tau_m^0(t))), \quad h(x) = x(0) - (1-\mu)x(\ell) + \mu c,$$
$$p_0(x)(t) = P(t)x(t), \quad \ell_0(x) = x(0) - (1-\mu)x(\ell).$$

Then evidently, the operators f, h, p_0, and ℓ_0 satisfy the above-mentioned conditions of Corollary 2 [6], and, in addition, $\|p_0(x)(t)\| \le \alpha(t)\|x\|_C$ for almost all $t \in I$ and every $x \in C(I; R^n)$, and $\|\ell_0(x)\| \le \alpha_0\|x\|_C$ for $x \in C(I; R^n)$, where $\alpha(t) = \|P(t)\|$ for $t \in I$ and $\alpha_0 = (2-\mu)$. Furthermore, the problem (2.1), (2.2), and consequently the problem (3.1) have only the trivial solution. Finally, every solution x to the problem (2.3), (2.4), and consequently to the problem (3.2), where $\lambda \in [0, 1]$, admits the estimate (2.5).

According to Corollary 2 [6], the problem (3.3) has at least one solution, and, consequently, the problem (1.3), (1.2) has also at least one solution. Since the problems (1.3), (1.2), and (1.1), (1.2), (1.2′) are equivalent, the theorem is valid.

Proof of Theorem 2.2. According to Theorem 2.1. we will show that under the assumption of Theorem 2.2 the problem (2.1), (2.2) has only the trivial solution, and that there exists such $\rho > 0$ that every solution x to the problem (2.3), (2.4) for $\lambda \in [0, 1]$ admits the estimate (2.5).

In our case the problem (2.1), (2.2) has the form

$$\frac{dx(t)}{dt} = diag(-p(t))x(t), \quad x(0) = (1-\mu)x(\ell).$$

The solution to this problem is a vector-valued function $x(t) = (c_{0i}\exp(-\int_0^t p(s)ds))_{i=1}^n$, where $c_{0i} \in R$ $(i = 1, \dots, n)$, satisfying the given boundary condition. However, from the assumption of theorem (see (2.8)) it follows that $c_{0i} = 0$ $(i = 1, \dots, n)$, and consequently, the problem (2.1), (2.2) has only the trivial solution.

Now let $\lambda \in [0, 1]$ and x be an arbitrary solution to the problem (2.3), (2.4). Then in view of (2.6) we have

$$diag(sgn x((t)))\frac{dx(t)}{dt} \le diag(-p(t))|x(t)| + P(t)|x(t)| +$$
$$+ \sum_{j=1}^m P_j(t)\chi_I(\tau_j(t))|x(\tau_j^0(t))| + \omega_0(t)$$

where $\omega_0 \in L(I; R_+^n)$.

The integration of the last inequality results in

$$|x(t)| \le diag\left(\exp\left(-\int_0^t p(s)ds\right)\right)|x(0)| +$$

$$+ \int_0^t P(s) diag\left(\exp\left(-\int_s^t p(\xi)d\xi\right)\right)|x(s)|ds +$$

$$+ \sum_{j=1}^m \int_0^t P_j(s)\chi_j(\tau_j(s)) diag\left(\exp\left(-\int_s^t p(\xi)d\xi\right)\right)|x(\tau_j^0(s))|ds + |\omega_0|_L$$

Hence, using the boundary conditions, applying the norm $|x|_C$ and Hölder's inequality we obtain

$$|x|_C \le S|x|_C + |\omega_0|_L,$$

where S is the matrix defined by (2.7). Since the spectral radius of S is strictly less than one, there exists an inverse matrix to the matrix $E - S$, and from the last inequality it follows that the estimate (2.5) holds, where $\rho = \|(E-S)^{-1}|\omega_0|_L\|$.

Proof of Theorem 2.3. Put $y = y_1 \dots = y_m = 0$. Then from the inequality (2.9) in Theorem 2.3 it follows that the inequality (2.6) in Theorem 2.2 is fulfilled, where $\omega(t) = f(t, 0, \dots, 0) \in L(I; R_+^n)$. Therefore, according to Theorem 2.2, the boundary value problem (1.1), (1.2), (1.2′) has at least one solution.

Suppose now that the considered problem has two solutions x and y. Denote by $z(t) = x(t) - y(t)$ for $t \in I$. From the inequality (2.9) it follows that

$$diag(sgn(z(t)))\frac{dz(t)}{dt} \le diag(-p(t))|z(t)| + P(t)|z(t)| +$$

$$+ \sum_{j=1}^m P_j(t)\chi_j(\tau_j(t))|z(\tau_j^0(t))|$$

and analogously to the proof of Theorem 2.2 we have

$$|z|_C \le S|z|_C.$$

Hence, in view of the properties of the matrix S defined by (2.7), it follows that

$|z|_C = 0$ and consequently, $x(t) = y(t)$ for $t \in I$.

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