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BASIS OF REGULARLY SHIFTED FUNCTIONS

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1. Introduction and Preliminaries

Sampling theorem for bandlimited signals [1], [2], as the backbone, together with digital technology progress, allowed bringing new quality to communication networks. Now TDM (Time Division Multiplexing) is well established and broadly used method for the transmission of multiple signals over common communication channels. The channel capacity is divided in time slots (for example 125 μs for telephone signal), and during a slot the sample value is transmitted. On the receiving side there is the continuous time signal reconstructed from received samples using analog or digital filters.

However the main challenge for today's communication networks is an integration of TDM and packet switched networks towards next generation networks (NGN). This brings a need of the sampling theorem generalization. Regarding deterministic signals, we can see three main streams of such generalization: wavelets [3], [4], shift invariant spaces [5] and parametric signals [6], [7].

The aim of this paper is to explain the principle ideas of the general sampling required for a voice-over-packet transmission. The explanation is supported by original proves.

More precisely; let $f(t) \in L_2(-\infty, \infty)$ be the Ω-bandlimited function i.e. there exists $\widehat{f}(\omega)$ such, that $\widehat{f}(\omega) = 0$ for $|\omega| \geq \Omega > 0$

where $\widehat{f}(\omega)$ is the Fourier transform of function $f(t)$:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}\, t$$

Then for $f(t)$ the sampling theorem is applied

$$f(t) = \sum_{k \in Z} f(k\Delta) \operatorname{sinc}(\Omega(t - k\Delta)), \qquad t \in R, \quad \Delta = \frac{\pi}{\Omega}$$

where

$$\begin{aligned} \operatorname{sinc}(t) &= \frac{\sin t}{t} \qquad t \in R \setminus \{0\} \\ &= \;\; 1 \qquad\quad t = 0 \end{aligned}$$

In other words the set $\{\Phi_k(t) = \operatorname{sinc}(\Omega(t - k\Delta)), \quad t \in R, \quad k \in Z\}$ creates the orthogonal basis of the Hilbert space $S \subset L_2(-\infty, \infty)$ of all Ω-bandlimited functions.

Now, when digital technology allows real time processing of voice and video signals, more sophisticated methods can be used. The main advantage of these methods is decreasing of the necessary channel capacity for the same quality signal transmission. In [8] it is shown that band limited functions with a non-compact spectrum can be transmitted with a sample rate lower than 2Ω, but more complex reconstruction procedure is necessary.

In both cases the subspace $S \subset L_2(-\infty, \infty)$ of band limited functions has the orthogonal basis done by a countable set of shifted functions

$$\{\Phi_k(t) = \Phi\ (t - k\Delta) \quad t \in R, \quad k \in Z\}$$

and the coordinates corresponding to this basis equal precisely the samples $f(k\Delta), \quad k \in Z$. It is very good property for realization to have basis of shifted functions, because as the time flows, the same generator creates new basis functions. We would like to keep this property.

But if digital technology allowed real time computation during the sample interval, coordinates of the function might be obtained by more general procedure than the sampling.

In this paper we would like to find several properties of the set

$$\{\Phi_k(t) = \Phi\ (t - k\Delta) \quad t \in R, \quad k \in Z\}$$

which creates a basis of a subspace $S \subset L_2(-\infty,\infty)$. Definitions and theorem in this section can be found for instance in [9]. We can meet it in the modern theory for basis and frames in Hilbert spaces.

Definition 1.1: *A system* $\{\Phi_k\} \subset L_2(-\infty,\infty),\ k \in Z$ *is called a Schauder basis in* $L_2(-\infty,\infty)$ *if for every* $f \in L_2(-\infty,\infty)$ *there exists a unique sequence* $\{c_k\}$ *such that* $f = \sum_{k \in Z} c_k \Phi_k$.

Definition 1.2: *A Schauder basis* $\{\Phi_k\} \subset L_2(-\infty,\infty),\ k \in Z$ *is called a Riesz basis in* $L_2(-\infty,\infty)$ *when the series* $\sum_{k \in Z} c_k \Phi_k$ *with coefficients* c_k *converges in* $L_2(-\infty,\infty)$ *if and only if* $\sum_{k \in Z} c_k < \infty$.

Definition 1.3*: A system of the functions* $\{\Phi_k\} \subset L_2(-\infty,\infty),\quad k \in Z$ *is called* ω *-independent if and only if*

$$\sum_{k \in Z} c_k \Phi_k = \theta \ \Rightarrow\ c_k = 0 \ \ \forall k \in Z$$

Definition 1.4: *A system of the functions* $\{\Phi_k\} \subset L_2(-\infty,\infty),\quad k \in Z$ *is called a frame if there exists constants* $0 < A \le B < \infty$ *such, that*

$$A \| f \|^2 \le \sum_{k \in Z} \left(f, \Phi_k \right)|^2 \le B \| f \|^2, \ \ \forall f \in L_2(-\infty,\infty)$$

Following theorem explains why the ω-independent of system $\{\Phi_k\}$ is important.

Theorem 1.5: *Let system of the functions* $\{\Phi_k\}$ *be a frame in* $L_2(-\infty,\infty)$*. Then* $\{\Phi_k\}$ *is a Riesz basis if only if* $\{\Phi_k\}$ *is* ω *-independent.*

2. Conditions for ω -independent Property in Spectral Domain

It is useful in the communication theory to study spectral properties of the functions. The following theorem shows a necessary and sufficient condition for $\Phi(t)$ to generate the ω-independent system of the functions.

Theorem 2.1: *Let* $\Delta > 0$ *and function* $\Phi \in L_2(-\infty,\infty)$*. The function* Φ *generates the* ω*-independent system* $\{\Phi_k\}$ *if and only if*

$$\sum_{n\in Z}\left|\hat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right| \neq 0 \quad \omega \in \left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right) \text{ a. e.} \tag{1}$$

where

$$\left\{\Phi_k(t) = \Phi\ (t-k\Delta) \subset L_2(-\infty,\infty) \quad t\in R, \quad k \in Z\right\}$$

$$\hat{\Phi}(\omega) = \int_{-\infty}^{\infty}\Phi(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t, \quad \omega \in R$$

Proof:

The functions $\Phi_k(t) = \Phi(t-k\Delta) \quad t\in R, \quad k\in Z$ are ω-independent if and only if

$$\sum_{k\in Z} c_k\Phi_k = \theta \quad \Rightarrow \quad c_k = 0 \quad \forall k \in Z$$

i.e. for almost every $t\in R$

$$\sum_{k\in Z} c_k\Phi_k(t) = \sum_{k\in Z} c_k\Phi(t-k\Delta) = 0 \quad \Rightarrow \quad c_k = 0 \quad \forall k\in Z \tag{2}$$

Let us apply Fourier transform to the left side of the implication (2).

Then we get

$$\hat{\Phi}(\omega)\sum_{k\in Z} c_k\,\mathrm{e}^{-\mathrm{j}\omega k\Delta} = \theta \quad \Rightarrow \quad c_k = 0, \quad \forall k\in Z$$

The sum $\sum_k c_k\,\mathrm{e}^{-\mathrm{j}\omega k\Delta}$ can be rewritten $\sum_n d_n\,\mathrm{e}^{\mathrm{j}\omega n\Delta}$, where $n=-k$ and $d_n = c_{-k}$.

The sum $\sum_n d_n\,\mathrm{e}^{\mathrm{j}\omega n\Delta}$ is the Fourier series of a periodical function $f \in L_2(-\infty,\infty)$

with the period $\dfrac{2\pi}{\Delta}$ (i.e. $f(\omega) = \sum_{n\in Z} d_n\,\mathrm{e}^{\mathrm{j}\omega n\Delta}$).

Then (2) is true if and only if for every periodical function $f\in L_2(-\infty,\infty)$ with period $\dfrac{2\pi}{\Delta}$ holds:

$$\hat{\Phi}\,f = \theta \quad \Rightarrow \quad d_n = 0$$

It is equivalent to

$$\widehat{\Phi} f = \theta \quad \Rightarrow \quad f = \theta \tag{3}$$

Due to periodicity of $f(\omega)$, implication (3) may be rewritten to

$$f(\omega)\sum_{n\in Z}\left|\widehat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right| = 0 \quad \Rightarrow \quad f(\omega)=0, \quad \omega\in\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right) \text{ a.e.} \tag{4}$$

Sufficient condition:

If $\Phi_k(t)$ are ω-independent, implication (4) is valid for all $f(\omega)\in L_2\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right)$. Then it has to be valid for a function $f_0 \in L_2\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right)$ such, that $f_0 \neq \theta$, i.e. $f_0(\omega)\neq 0 \qquad \omega\in\left\langle-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right\rangle$.

Due to (4), it has to be valid that

$$f_0(\omega)\sum_{n\in Z}\left|\widehat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right| \neq 0 \quad \omega\in\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right) \text{a.e.}$$

Therefore

$$\sum_{n\in Z}\left|\widehat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right| \neq 0 \quad \omega\in\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right) \text{a.e.}$$

Necessary condition:

If $\sum_{n\in Z}\left|\widehat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right| \neq 0 \quad \omega\in\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right)$ a.e.

then

$$f(\omega)\sum_{n\in Z}\left|\widehat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right| = 0 \quad \Rightarrow \quad f(\omega)=0, \quad \omega\in\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right) \text{ a.e.}$$

Last implication is equivalent to the definition of the ω-independent of the system $\{\Phi_k, k\in Z\}$.

Following examples illustrate the condition proved in theorem 2.1.

Example 2.2: *Function* $\Phi(t) = \mathrm{e}^{-|t|}$ *creates the* ω*-independent system of functions* $\Phi_k = \mathrm{e}^{-|t-k\Delta|}$,

because $\hat{\Phi}(\omega) = \dfrac{2}{1+\omega^2} > 0 \quad \forall \omega \in \left(-\dfrac{\pi}{\Delta}, \dfrac{\pi}{\Delta}\right)$

In fact $\sum_{k \in Z} c_k \, \mathrm{e}^{-|t-k\Delta|} = 0 \quad \Rightarrow \quad c_k = 0 \quad \forall k \in Z$

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Example 2.3: *Function* $\Phi(t) = \operatorname{sinc}(\Omega t)$ *where* $\Omega = \dfrac{\pi}{2\Delta}$ *does not create the* ω*-independent system* $\Phi_k(t) = \operatorname{sinc}\left(\dfrac{\pi}{2\Delta}(t - k\Delta)\right)$.

Fourier transform of the function $\Phi(t)$ *is* $\hat{\Phi}(\omega) = 2\Delta$ for $\omega \in \left(-\dfrac{\pi}{2\Delta}, \dfrac{\pi}{2\Delta}\right)$

and $\hat{\Phi}(\omega) = 0$ for $\omega \in R \setminus \left\langle \left(-\dfrac{\pi}{2\Delta}, \dfrac{\pi}{2\Delta}\right) \right\rangle$. So $\sum_{n \in Z} \left| \hat{\Phi}\left(\omega + \dfrac{2\pi n}{\Delta}\right) \right| = 0$

for $\omega \in \left(-\dfrac{\pi}{2\Delta}, \dfrac{\pi}{2\Delta}\right)$ and according theorem 2.1. the system $\{\Phi_k, k \in Z\}$ is not ω -independent.

Really the function $\Phi_1(t) = \operatorname{sinc}\left(\dfrac{\pi}{2\Delta}(t - \Delta)\right)$ is $\dfrac{\pi}{2\Delta}$-bandlimited

and we can write

$$\Phi_1(t) = \sum_{k \in Z} \Phi(2k\Delta) \operatorname{sinc}\left(\frac{\pi}{2\Delta}(t - 2k\Delta)\right) = \sum_{k \in Z} \Phi(2k\Delta) \Phi_{2k}(t)$$

it implied $\quad 0 = \Phi_1(t) - \sum_{k \in Z} \Phi(2k\Delta) \Phi_{2k}(t) = \sum_{m \in Z} d_m \Phi_m(t)$

but $d_1 \neq 0$ i.e. system $\{\Phi_k, k \in Z\}$ is not ω-independent.

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3. Conditions for the Orthogonality of System $\{\Phi_k, k \in Z\}$

The procedure of coordinate calculation is simpler, if $\Phi(t)$ generates the orthogonal basis.

$$(\Phi_k, \Phi_l) = \int_R \Phi(t - k\Delta)\Phi(t - l\Delta)dt = 0, \quad \forall k, l \in Z; \quad k \neq l$$

The equivalent condition for $\Phi_k(t) = \Phi(t - k\Delta) \quad t \in R, \quad k \in Z$ is

$$(\Phi_k, \Phi_0) = \int_R \Phi(t - k\Delta)\Phi(t)dt = 0, \quad \forall k \in Z; \quad k \neq 0$$

In the following text we suppose orthonormal basis with

$$(\Phi_k, \Phi_k) = \int_R \Phi^2(t)dt = 1, \quad \forall k \in Z$$

If the function f satisfies $f = \sum_{k \in Z} c_k \Phi_k$ then coefficients c_k can be calculated as

$$c_k = (f, \Phi_k) = \int_R f(t)\Phi(t - k\Delta)\, dt, \quad \forall k \in Z$$

Theorem 3.1: *Let* $\Delta > 0$ *and function* $\Phi \in L_2(-\infty, \infty)$. *Function* $\Phi(t)$ *generates the orthonormal basis* $\{\Phi_k(t) = \Phi(t - k\Delta), \ t \in R, \ k \in Z\}$ *if and only if*

$$\sum_{n \in Z} \left| \widehat{\Phi}\left(\omega + \frac{2\pi n}{\Delta}\right) \right|^2 = \Delta \qquad \omega \in \left(-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}\right) \tag{5}$$

Proof:

The basis is orthonormal if and only if the following holds

$$(\Phi_k, \Phi_0) = \int_R \Phi(t - k\Delta)\Phi(t)dt = \begin{cases} 1 & k=0 \\ 0 & \forall k \in Z; \quad k \neq 0 \end{cases} \tag{6}$$

Let $\Phi^*(t)$ be a convolution $\Phi(t)$ with $\Phi(-t)$

$$\Phi^*(t) = \int_R \Phi(\tau)\Phi(\tau - t)dt$$

Fourier transform of convolution is

$$\widehat{\Phi}^*(\omega) = \int_R \Phi^*(t) e^{-j\omega t} dt = \left|\widehat{\Phi}(\omega)\right|^2 \qquad \omega \in R$$

Then relation (6) can be rewritten as

$$\left(\Phi_k,\Phi_0\right)=\int_R \Phi(t-k\Delta)\Phi(t)dt=\Phi^*(k\Delta)=\begin{cases}1 & k=0\\ 0 & \forall k\in Z;\quad k\neq 0\end{cases}$$

or by using inverse Fourier transform

$$\left(\Phi_k,\Phi_0\right)=\frac{1}{2\pi}\int_R \left|\hat{\Phi}(\omega)\right|^2 \mathrm{e}^{j\omega k\Delta}d\omega=\begin{cases}1 & k=0\\ 0 & \forall k\in Z;\quad k\neq 0\end{cases}$$

Dividing R to intervals with length $\frac{2\pi}{\Delta}$, we can write

$$\frac{1}{2\pi}\sum_{n\in Z}\int_{-\frac{\pi}{\Delta}}^{\frac{\pi}{\Delta}}\left|\hat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right|^2 \mathrm{e}^{j\omega k\Delta}d\omega=\begin{cases}1 & k=0\\ 0 & \forall k\in Z;\quad k\neq 0\end{cases}$$

Let us change order of summation and integration and denote

$$S(\omega)=\sum_{n\in Z}\left|\hat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right|^2$$

we obtain

$$\frac{1}{2\pi}\int_{-\frac{\pi}{\Delta}}^{\frac{\pi}{\Delta}}S(\omega)\mathrm{e}^{j\omega k\Delta}d\omega=\begin{cases}1 & k=0\\ 0 & \forall k\in Z;\quad k\neq 0\end{cases}\qquad (7)$$

Coordinates of the Fourier series of the function $S(\omega)$ can be calculated as

$$c_k=\frac{\Delta}{2\pi}\int_{-\frac{\pi}{\Delta}}^{\frac{\pi}{\Delta}}S(\omega)\mathrm{e}^{-j\omega k\Delta}d\omega$$

and comparing c_k to (7), we can say that all of the coordinates except the average value ($k=0$) equal zero and $c_0=\Delta$. Then

$$c_0=\sum_{n\in Z}\left|\hat{\Phi}\left(\omega+\frac{2\pi n}{\Delta}\right)\right|^2=\Delta\qquad \omega\in\left(-\frac{\pi}{\Delta},\frac{\pi}{\Delta}\right)$$

Example 3.2.: *Any function* $\Phi \in L_2(-\infty,\infty)$, *such that* $\int_R \left|\widehat{\Phi}(\omega)\right|^2 dt = 1$ *and* $\Phi(t) = 0 \ \ |t| > \frac{\Delta}{2} > 0$ *generates orthonormal basis*

$$\left\{\Phi_k(t) = \Phi(t - k\Delta), \ \ t \in R, \ \ k \in Z\right\},$$

$$\left(\Phi_0, \Phi_k\right) = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \Phi(t)\Phi(t - k\Delta)dt = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

because of the definition the theorem 3.1. then implies that for these functions it holds

$$c_0 = \sum_{n \in Z} \left|\widehat{\Phi}\left(\omega + \frac{2\pi n}{\Delta}\right)\right|^2 = \Delta \qquad \omega \in \left(-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}\right).$$

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Summary

BASIS OF REGULARLY SHIFTED FUNCTIONS

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We have explained the basic ideas of the generalized sampling, which can be used for the voice-over-packet transmission if packet loss can occur. Basic properties of the basis generating signal in the spectral domain are given and proved. The paper explains when the set of regularly shifted functions creates a general Riesz basis or an orthogonal basis. Basis examples support an easy view to this structure.

Resumé

BÁZE TVOŘENÉ PRAVIDELNÝM POSOUVÁNÍM FUNKCÍ

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Článek vysvětluje základní myšlenky zevšeobecněného vzorkování, které se dá použít pro přenos hlasu pomocí paketů, jestliže se může vyskytnout ztráta paketů. Vysvětleny a dokázány jsou základní vlastnosti které má mít signál v spektrální oblasti, jestliže má generovat bázi spolu se signály, které vzniknou jeho rovnoměrným posouváním v čase. Článek vysvětluje tyto vlastnosti jak pro všeobecnou Rieszovu bázi, stejně tak pro bázi ortogonální. Pro názornější výklad jsou v článku uvedeny příklady funkcí s požadovanými vlastnostmi.

Zusammenfassung

BASEN GEBILDET MIT DER REGULÄREN FUNCTIONENVERSCHIEBUNG

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In diesem Artikel verden fundamentale Ideen der verallgemienesten Abtastung dasgestellt. Grundlegende Eigenschaften des basiserzeugenden Signals werden im Spektralberuch erläuten und beweisen. Der Artikel zeigt auf unter welchen Voraussetzungen die regular verschobenen Funktionen eine Riesz-Basis sovie eine ortogonale Basis erzeugen. Einige Beispiele verdeutlichen diesen Sachverhalt. Diese Abtastung kann zur Sprachpaketubertragung mit moglichen Paketverlust augewendet werden.