Proceedings of 26th International Scientific Conference. Transport Means. 2022.

The Position of Eigenvalues in the Gaussian Complex Plane Depending on the Change of the Coefficients of the Homogeneous Linear Differential Equation in Transport Application Using Matlab

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Abstract

The mathematical solution of vibration of a single-degree-of-freedom dynamical system always leads to the construction and solution of a second-order linear ordinary differential equation with constant coefficients. The coefficients of this equation correspond to the mass of the body, the damping coefficient of the damper, and the stiffness of the spring in a given system. The paper examines how changes of these coefficients influence the position of eigenvalues in the Gaussian complex plane. For the eigenvalues of the second-order homogeneous linear differential equation, it is derived and proved that the product of their distances from the origin of the Gaussian complex plane is constant and equal to the numerical value of the natural circular frequency of the corresponding mass-damper-spring system. It is further shown and proved that these eigenvalues follow the rules of conformal mapping of circular inversion with respect to a reference circle with its center at the origin of the Gaussian complex plane and a radius equal to the square root of the natural circular frequency of the corresponding system. Furthermore, third and higher order homogeneous linear differential equations are also investigated, and a similar property is derived and proved, namely that the product of the absolute values of the eigenvalues is linearly dependent on the coefficients of the differential equation. This issue has been used in the teaching of the applied mathematics and numerical methods in transport at the Faculty of Transport Engineering, University of Pardubice.

KEY WORDS: *Applied Mathematics in Transport, Linear differential equation, single-degree-of-freedom dynamical system, eigenvalues, circular inversion*

1. Introduction

The study of changes in physical quantities with respect to time, such as temperature, pressure, deflection, velocity, stress, concentration, etc., can be realized by simulation using differential equations. It is important for engineers to be able to model technical problems using mathematical equations and then solve these equations so as to make it possible to study the behavior of the corresponding systems based on the obtained results. Differential equations and their systems are an essential part of mathematical analysis. They make it possible to solve a wide range of practical technical tasks. The theory of solving differential equations and their systems is described in detail, e.g., in [1,4,5,7,8]. This article deals with an issue which is only rarely discussed in the theory of solving differential equations or in the theory of single or higher degree-of-freedom dynamical systems – the position of eigenvalues in the Gaussian complex plane depending on the change of the constant coefficients of the differential equation.

2. Motivation model

Modeling of single-degree-of-freedom dynamical systems led us to the idea of examining the relationships between the eigenvalues of homogeneous linear differential equations (HLDE).

2.1. The response of a single-degree-of-freedom dynamical system and its conversion

A single-degree-of-freedom dynamical system can be modeled using the mass-damper-spring mechanical system. From [1,3,6,9,10,11] we will get the equation of motion for a single-degree-of-freedom dynamical system $m\ddot{x} + b\dot{x} + kx = F(t)$ where $F(t)$ is a known forcing function (its equal to zero - unforced oscillations), m is the mass of the body, b is the damping coefficient of the viscous proportional damper, k is the stiffness of the spring, x is the deflection and $\dot{x}$ and $\ddot{x}$ are the first and second derivatives of the deflection with respect to time.

This is a second order HLDE with constant coefficients, which may be converted into

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

2.2. M-file for calculation and graphic representation of eigenvalues

Calculation and graphical representation of eigenvalues was performed using the script below. The % character is always followed by a commentary describing the current stage of the calculation.

```
% input data setup
m=10;
k=1000;
b=linspace(0,500,51);
% computation of eigenvalues
n=length(b);
for i=1:n
    S(:,i)=eig([0 1; -k/m -b(i)/m]);
    AHVl_c(:,i)=abs(S(:,i));
    KI(i)=AHVl_c(1,i)*AHVl_c(2,i);
end

% extraction of results
S
V=[AHVl_c ;KI]
r=AHVl_c(1)
p=KI(end)
x=max(max(abs(real(S))));
y=max(max(abs(imag(S))));

% graphical representation of the
results
set(gcf,'WindowState','Maximized');
```

```
subplot(4,1,1)
plot(b,KI,'*')
grid on
title('a)','FontSize',12)
axis([-2 502 88 112]);
yticks([90 100 110])

subplot(4,1,[2 3 4])


col={'b' 'g' 'r' 'c' 'm' 'y' 'k'}; % every pair of the eigenvalues use other colour
for i=1:n
          b=mod(i,7)+1;
          plot(real(S(1,i)),imag(S(1,i)),'*','color',col{b},'linewidth',4,…
          'markersize',6);
          hold on;
          plot(real(S(2,i)),imag(S(2,i)),'*','color',col{b},'linewidth',4,…
          'markersize',6);
          set(gca,'YAxisLocation','origin');
          set(gca,'XAxisLocation','origin');
          axis equal
          axis([-x-1 2 -y-0.5 y+0.5]);
          xlabel('Real axis','Position',[-x+5 0.3 1']);
          ylabel('Imaginar axis','Position',[-0.5 4 1']);
          title('b)','FontSize',12);
          grid on;
     drawnow;
end
```

3. Numerical results

We will apply the computational model to selected input data, which in our case are coefficients k, m and b. The conditions for these coefficients are determined according to technical practice $k, m, b \in R^+$. In each experiment, parameters k, m will be fixed, because $\frac{k}{m} = \omega_0^2$ and their ratio plays a role. Parameter b will always be changed in a given interval with a given number of subintervals p, because its value is given by the relationship $b = 2b_k\sqrt{km}$.

3.1. Example 1

In example 1, let us choose $k = 2200\ \left[\frac{N}{m}\right]$, $m = 19{,}572\ [kg]$, $p = 51$ and $b \in \langle 0; 1000\rangle [N \cdot sec \cdot m^{-1}]$. We will calculate eigenvalues for each value of b and plot them in the Gaussian complex plane. The results are shown in Figure 1. The eigenvalues are either two negative real numbers, one double real or two complex conjugate numbers, which lie on a semicircle with radius $\omega_0 = \sqrt{\frac{2200}{19.57}} \doteq 10.6$ with the center at the origin. The product of the absolute values of the eigenvalues is always equal to the value $\omega_0^2 = \frac{2200}{19.57} \doteq 112.4$.

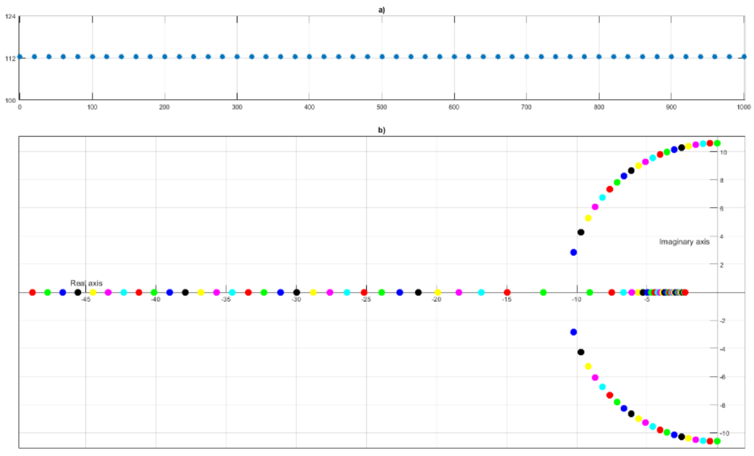

Figure 1: Eigenvalues from Example 1 a) product of the absolute values of the eigenvalues b) position of eigenvalues in Gaussian complex plane

4. Conclusions of mechanical and numerical analysis of a single-degree-of-freedom dynamical system

From the results presented above, we can see that the eigenvalues always lie either on the real axis or on a circle, and the product of their absolute values is constant. We will generalize the whole issue and propose a hypothesis, which we will prove later. First, lets us recall the definition of circular inversion. Circular inversion, as shown in Figure 2, determined by a reference circle $\omega(S, r)$, is an involuntary and conformal mapping, which to each point $X \neq S$ assigns point X', as follows:

$$1. X' \in \rightarrow SX \text{ and } 2. |SX| \cdot |SX'| = r^2. \tag{2}$$

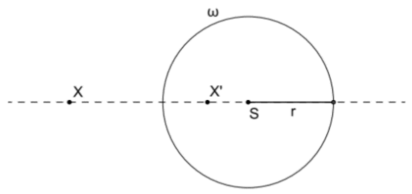

Figure 2: Circular inversion is an involuntary mapping, which means the image of point X′ is point X

4.1. Hypothesis

Let us consider any linear homogeneous second-order HLDE with constant coefficients

$$a_1\ddot{x} + a_2\dot{x} + a_3 x = 0, \tag{3}$$

where the coefficients a_1, a_2, a_3 correspond to the coefficients in Equation (3), therefore $a_1 = 1$, $a_2 = \frac{b}{m}$ and $a_3 = \frac{k}{m}$, and which can be any real number. We will only exclude the possibility of $m = 0$. The product of the absolute values of the eigenvalues is then constant and its numerical value is equal to coefficient a_3, which, in the case of a single-degree-of-freedom dynamical system, corresponds to the square of the natural circle frequency of the undamped system. Furthermore, the distribution of the eigenvalues in the Gaussian complex plane is controlled by a circular inversion with a reference circle having a center at the origin and a radius equal to the square root of the product of the absolute values of the corresponding eigenvalues, which, again, is the value corresponding to the natural circle frequency of the undamped system.

4.2. Proof

We can approach the whole issue in terms of Viet formulas [13], i.e., relations between the roots and coefficients of a quadratic equation. The characteristic equation $\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$, has either two real solutions $\gamma, \delta \in R$, which do not necessarily have to be different, or two complex conjugates $\alpha + \beta i, \alpha - \beta i \in C$. When we perform root factors decomposition $(\lambda - \gamma)(\lambda - \delta) = \lambda^2 - (\gamma + \delta)\lambda + \gamma\delta = \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}$, it follows that $\gamma\delta = \frac{k}{m} = \omega_0^2$, or $(\lambda - \alpha + \beta i)(\lambda - \alpha - \beta i) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}$, and this implies $\alpha^2 + \beta^2 = (\alpha + \beta i)(\alpha - \beta i) = \frac{k}{m} = \omega_0^2$. The roots lie on a circle with radius ω_0, as follows from the relationship $|\alpha + \beta i| = \sqrt{\alpha^2 + \beta^2} = \omega_0$.

5. HLDE of higher orders with constant coefficients

So far, we have considered only second-order linear homogeneous differential equations with constant coefficients. Below, we will demonstrate further similarities regarding differential equations of the third and higher order. Let us consider a third-order linear homogeneous differential equation

$$a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0, \tag{4}$$

with constant coefficients a, b, c, d, where $a \neq 0$. We can construct the corresponding characteristic equation $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic equation. We can calculate the product of their absolute values and the result is same in all possible cases $\left|2q + \frac{b^3}{27a^3} + p\frac{b}{a}\right|$. If we further replace $\frac{3ac-b^2}{9a^2}$ and $\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}$ with p and q, see (13), and simplify, we get $2q + \frac{b^3}{27a^3} + p\frac{b}{a} = \frac{2b^3}{27a^3} - \frac{2bc}{6a^2} + \frac{2d}{2a} + \frac{b^3}{27a^3} + \frac{3ac-b^2}{9a^2} \cdot \frac{b}{a} = \frac{2b^3+9abc+b^3-3b^3}{27a^3} - \frac{2bc}{6a^2} + \frac{d}{a} = \frac{bc}{3a^2} - \frac{bc}{3a^2} + \frac{d}{a} = \frac{d}{a}$, which is the absolute term in Equation (4) if we divide the whole equation by value a. This finding must be consistent with Vieta's formulas [13].

5.1. Example 2

In Example 2, for a third-order HLDE, we will firstly divide Equation (4) by coefficient a and choose $p = \frac{b}{a} = 1, q = \frac{c}{a} = 1$, and $r = \frac{d}{a}$ in the interval $\langle -100; 100 \rangle$ which we will divide into 51 parts. For each value r, we will calculate the eigenvalues and the product of their absolute values. The situation is shown in Figure 3 – part a) shows the product of absolute values of eigenvalues (we can see its linear dependence on the given parameters), part b) shows the corresponding eigenvalues connected in a triangle, and in part c) we can see the position of eigenvalues, and the curves along which the eigenvalues move are visible.

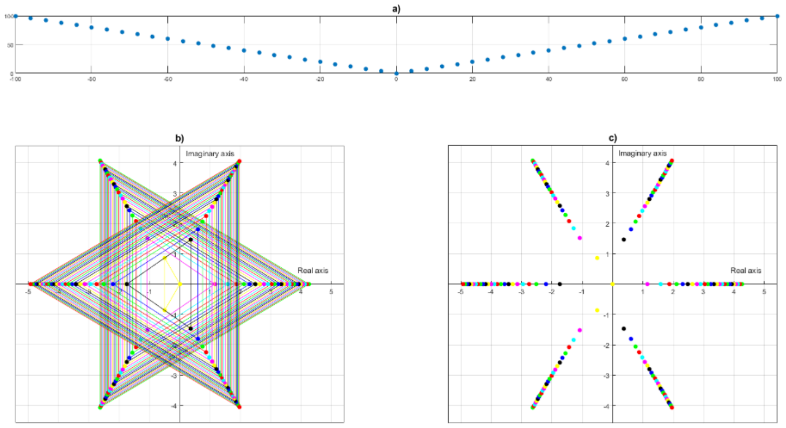

Figure 3: Eigenvalues from Example 2
a) Product of the absolute values of the eigenvalues
b) Position of eigenvalues in the Gaussian complex plane, corresponding eigenvalues are connected in a triangle
c) Position of eigenvalues in the Gaussian complex plane

5.2. Results for higher degrees

In Figures 4 and 5 we can see the results for fourth-order and fifth-order HLDE.

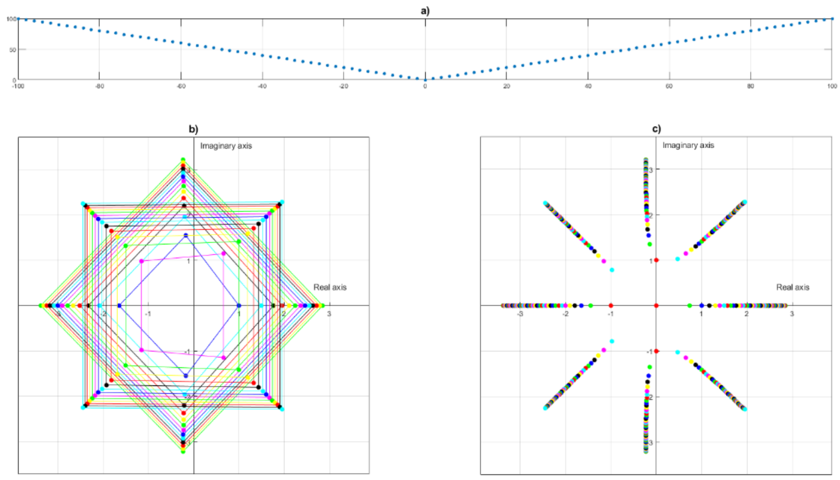

Figure 4: Eigenvalues from Example 4
a) Product of the absolute values of the eigenvalues
b) Position of eigenvalues in the Gaussian complex plane, corresponding eigenvalues are connected in a quadrilateral
c) Position of eigenvalues in the Gaussian complex plane

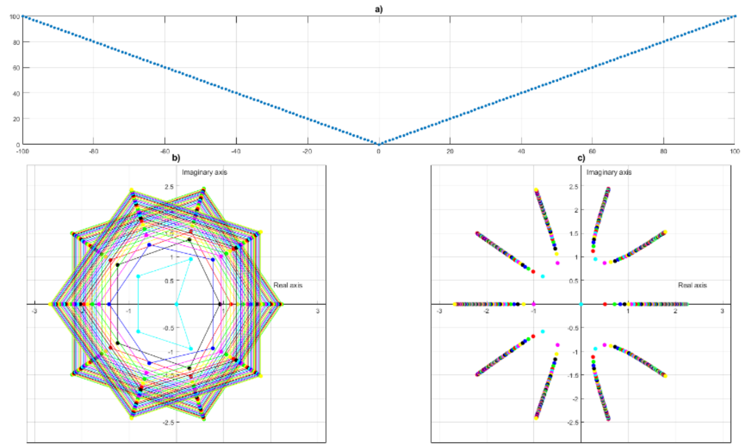

Figure 5: Eigenvalues from Example 5
a) Product of the absolute values of the eigenvalues
b) Position of eigenvalues in the Gaussian complex plane, corresponding eigenvalues are connected in a pentagon
c) Position of eigenvalues in the Gaussian complex plane

6. Discussion

If, for parameters m, k, and b, it holds that $b = \pm 2\sqrt{mk}$, we will always get one double eigenvalue which lies at the intersection of the real axis and the reference circle of the circular inversion. This case corresponds to a system with critical damping, where the system does not oscillate and is only damped. One double eigenvalue is in the position shown in Figure 6a. If, for parameters m, k, and b, it holds that $b \in \left(-2\sqrt{mk}; 2\sqrt{mk}\right)$, we always get two complex conjugate eigenvalues which lie on the reference circle of the circular inversion, and they are self-conjugate points. This case corresponds to a damped system, where the system oscillates with a decreasing amplitude. A special case occurs when the damping coefficient is equal to zero, then the oscillation is not damped and the amplitude remains the same. The eigenvalues are in the position shown in Figure 6b. If it holds that $c \in \left(-\infty; -2\sqrt{mk}\right) \cup \left(2\sqrt{mk}; +\infty\right)$, we will always get two real different eigenvalues which lie on the real axis and each is the image of the other in circular inversion with a reference circle having its center at the origin and radius equal to the square root of the natural frequency of the system $\omega_0^2 = \frac{k}{m}$. This case corresponds to an overdamped system that does not oscillate and the amplitude decreases exponentially. The eigenvalues are in the position shown in Figure 6c.

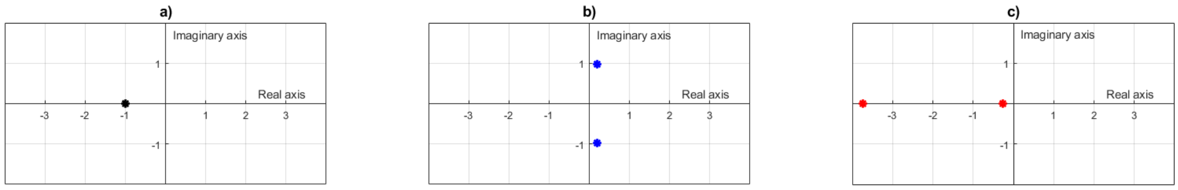

Figure 6: Position of eigenvalues for second-order linear homogenous differential equation in three basic cases

In the case of third-order HLDE, the corresponding eigenvalues are always one real and two complexly conjugate numbers, they form an equilateral triangle in the Gaussian plane of complex numbers and the vertices of the triangle move along lines or parabolas. Regarding HLDE of higher orders, the eigenvalues always form vertices of an n-gon.

7. Conclusion

Based on a numerical analysis, we expressed and proved the hypothesis that in a mechanical mass-damper-spring system, which is a single-degree-of-freedom dynamical system corresponding to the solution of a second-order HLDE with constant coefficients, the following statement holds. The product of the distances of the corresponding eigenvalues from the origin is constant and equal to the square of the natural circular frequency of the system. The eigenvalues satisfy the conditions of a circular inversion with respect to a reference circle with its center at the origin and radius equal to the natural frequency of the system. A similar property for eigenvalues was proved for third-order HLDE equation with constant coefficients, and it was additionally shown that the same property applies to eigenvalues for corresponding higher-order HLDE. Finally, a further direction of research on this issue was suggested.

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