

THE USE OF WEIBULL DISTRIBUTION FOR RELIABILITY ASSESSMENT

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1. INTRODUCTION

An important problem in reliability assessment is the determination of the probability, with which can be expected that a random variable y (e.g. strength or life time) does not attain a certain level, or, vice versa, determination of the „guaranteed minimum value“, corresponding to a very low probability of its non-exceedance. The value of the distribution function F is calculated in the former case, and the quantile in the latter. The distribution of strength or time to failure can often be approximated by the Weibull distribution function:

$$F(y) = 1 - \exp \left[- \left(\frac{y - \gamma}{\alpha} \right)^\beta \right] \quad \text{for } y > \gamma, \quad (1)$$
$$F(y) = 0 \quad \text{for } y \leq \gamma.$$

The quantile, corresponding to the probability F of non-exceedance, is determined as

$$y(F) = \gamma + \alpha \left[-\ln(1 - F) \right]^{1/\beta}. \quad (2)$$

Three constants, i.e. the parameter of the distribution shape β , the scale parameter α , and the location parameter γ (corresponding to the minimum possible or threshold value) give this distribution great flexibility, and, together with its simple form (1) and (2), contribute to its wide use. However, there are also problems. The rigorous procedure for the determination of parameters of a three-parameter distribution is complicated. Moreover, the quantities a , b , c , determined from the finite number of data, are only estimates of the true parameters α , β , γ .

Thus, also the calculated values of the distribution function $F(y)$ or quantiles $y(F)$ are only estimates, and their true values can be different. The determination of confidence limits for the parameters α, β only is not sufficient, since both quantities are correlated. The methods for the determination of the confidence interval for the threshold value γ are either not transparent (the use of special tables is necessary), or not very accurate. In reliability analysis, however, the knowledge of the minimum guaranteed value or low-probability quantiles is extremely important. This paper presents a simple method for estimation of the parameters α, β, γ , and the confidence limits for quantiles and the probability F .

2. DETERMINATION OF DISTRIBUTION PARAMETERS

The estimates a, b, c of the parameters α, β, γ can be obtained in more ways, the best known being the method of moments, method of maximum likelihood, and method of least squares. The first two methods need special tables or an iterative procedure; see e.g. [1, 2]. In engineering practice, therefore, the third method is often used, which also enables an easy visual check of the suitability of the approximation. This method uses the fact that a rearrangement of Eq. (1) gives the expression

$$\ln(y - c) = \ln a + \frac{1}{b} \ln \left(\ln \frac{1}{1 - F} \right), \quad (3)$$

which corresponds to the straight line equation

$$Y = A + BX \quad (4)$$

in coordinate system

$$Y = \ln(y - c), \quad X = \ln \left\{ \ln \left[1 / (1 - F) \right] \right\}, \quad (5)$$

with unknown constants $A (= \ln a)$, $B (= 1/b)$ and c . The way of their determining depends on the expected minimum value of y .

2. 1 Two-parameter distribution

Strength or time to failure are sometimes assumed to have the minimum possible value y_{\min} zero, so that $c = 0$. The determination of the remaining two parameters a, b in such case is easy. Ranking the measured values y_j in ascending order (the minimum value has subscript $j = 1$, the maximum value has $j = n$, where n is the number of measurements), and coordinating each value the probability F_j (see below), the corresponding values X_j, Y_j are calculated, from which the constants A, B are determined using the least squares method (Fig. 1), and then a, b .

The probability F_j , corresponding to the j -th value, is usually calculated using some of the following expressions:

$$F_j = \frac{j - 0.5}{n} \quad (a), \quad F_j = \frac{j - 0.3}{n + 0.4} \quad (b), \quad F_j = \frac{j}{n + 1} \quad (c). \quad (6)$$

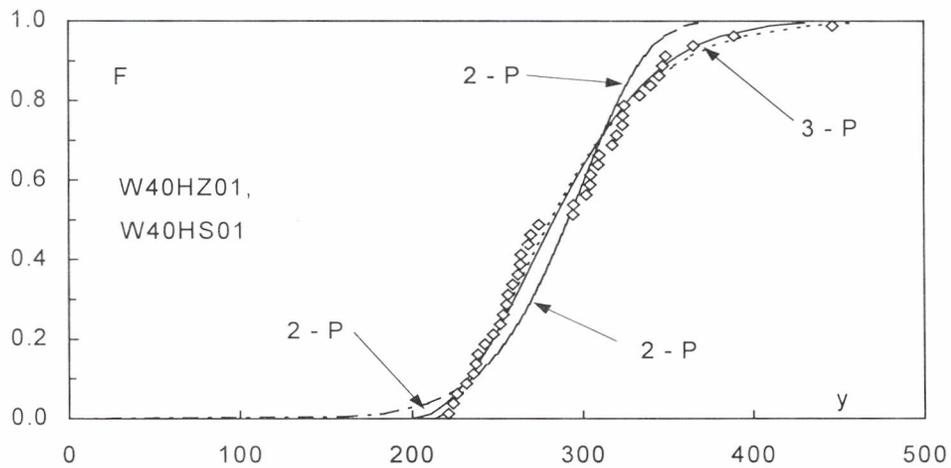
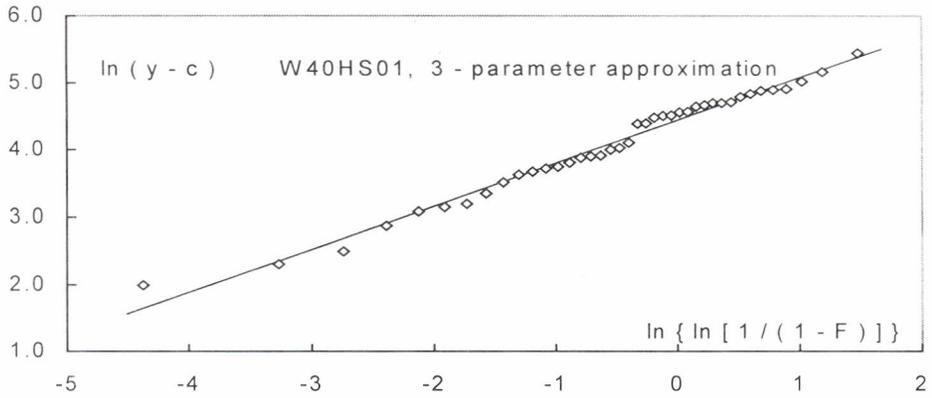
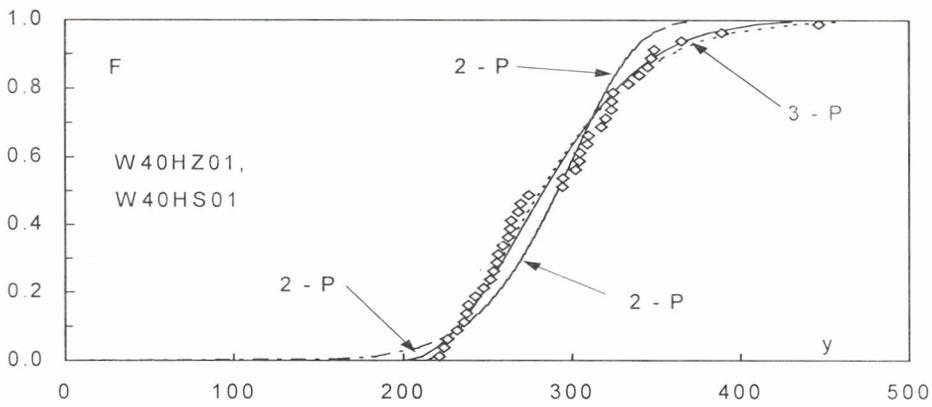


Fig. 1 Random sample W40H01 from the parent distribution with parameters $\alpha = 100$, $\beta = 2$, $\gamma = 200$, approximated by Weibull distribution. 2-P - two-parameter distribution ($c = 0$); 3-P - three-parameter distribution ($c = 213,9$; $y_1 = 206,3$; sample size $n = 40$).

Each expression gives somewhat different values F_j , and it is impossible to decide, which one is the best. The formula (a) according to Hazen gives the mean of the j -th interval for F , while the expression (b) corresponds approximately to the median of y in the j -th interval [3, 4]. The expression (c), recommended by Gumbel [5, 2] yields the highest F values for low j -values, so that the pertinent distribution function should give the most conservative results, i.e. higher probability F with which a certain very low value of y can occur, or lower value of a quantile y for the demanded low probability F . A comparison of expressions (a) - (c) also shows that the F_j values obtained using Eq. (6b) will always lie between those obtained by (6a) and (6c).

Notice. The F_j values have a definite value, given by the number of the measurements, n . Thus, X corresponds to the independent variable in the least squares method, while Y corresponds to the dependent variable.

2. 2 Three-parameter distribution

The lowest value of y is sometimes obviously higher than zero. (This is also indicated by the non-linear distribution of the measured values plotted for $c = 0$ in the $Y(X)$ diagram; cf. Fig. 1). The reliability assessment or determination of quantiles using two-parameter distribution could be too conservative (leading to poor utilising the strength of the material or life of the structure), and the three-parameter Weibull distribution may be more suitable. The threshold value c is sometimes determined empirically as the asymptote of the values $Y = \ln y$ plotted in the diagram $Y(X)$, or as the value, which gives the $Y(F)$ function the „best“ linear shape (Fig. 1). A less empirical approach utilises the condition that the scatter of the measured values around the distribution function must be minimum. This scatter can be characterised: a) by means of the sum of squared differences between the transformed measured values Y_{mj} and the corresponding values Y_j of the linearised regression function (4), or b) by means of the sum of squared differences between the measured values y_{mj} and the corresponding values y_j of the untransformed function (2). The method a) gives implicitly more weight to the lowest values y_j , since the logarithm of $y_j - c$ (i.e. the distance from the regression line) grows significantly for very small values of this difference. The method b), which minimises the scatter around the original (untransformed) distribution function, does not „prefer“ any value, and accounts better for the general character of the distribution.

With the approach b), it is possible to work either with the sum of squared differences or with the residual standard deviation, defined by

$$s_{res} = \sqrt{\frac{\sum_j (y_{mj} - y_j)^2}{\nu}} = \sqrt{\frac{\sum_j \left\{ y_{mj} - c - a \left[-\ln(1 - F_j) \right]^{1/b} \right\}^2}{n - 2}}; \quad (7)$$

The number of degrees of freedom ν is smaller than the number of measurements n by 2 (which correspond to two regression constants a, b ; the third constant (c) is chosen).

When the a) approach is used, which works with the linear regression function, it is also possible to use the residual standard deviation, defined by

$$S_{res} = \sqrt{\frac{\sum_j (Y_{mj} - Y_j)^2}{\nu}} = \sqrt{\frac{\sum_j (Y_{mj} - A - B X_j)^2}{n - 2}} \quad (8)$$

As a criterion in this case, the coefficient of correlation, r , may be used as well, and the fit of the transformed (linearised) data is best if r is maximum.

The procedure is as follows. A starting value c_1 is chosen (e.g. the minimum measured value of y), the corresponding values Y_{j1} and F_{j1} are determined using Eqs. (5) and (6), then (using the least squares method) the constants A_1, B_1 in the straight line equation (4), and the constants a_1, b_1 of the distribution function (1). Inserting a_1 and b_1 , together with c_1 , into Eq. (7) or (8) yields the residual standard deviation $s_{res,1}$ or $S_{res,1}$. In the following step, c_2 is chosen (a little smaller than c_1), and the values a_2, b_2 and $s_{res,2}$ (or $S_{res,2}$) are calculated and compared with $s_{res,1}$ (or $S_{res,1}$). This continues until the residual standard deviation is minimum. The corresponding values a, b, c are the best estimates of α, β, γ . (An analogous procedure is used if the optimisation criterion is the maximum of r .) A great advantage of this approach is a simple algorithm for finding the optimum values a, b, c ; they can also be quickly found using universal programs like Excel.

3. DETERMINATION OF CONFIDENCE INTERVALS FOR QUANTILES AND VALUES OF DISTRIBUTION FUNCTION

The quantiles $y(F)$ of the Weibull distribution are calculated using Eq. (2). Instead of the true parameters α, β, γ , however, only their estimates a, b, c are inserted into Eq. (2), which were obtained from a (random) sample of size n . The use of another sample would result in different values of a, b, c , as well as of quantiles $y(F)$. The procedures for determination of confidence limits for the parameters α, β of two-parameter distribution can be found in various literature; e.g. [2, 3]. However, there are only few works regarding the confidence interval for the threshold value γ of the three-parameter distribution, and the pertinent methods are either little transparent and need special tables [2], or are not much accurate [6]. In reliability analysis, however, it is just the minimum guaranteed value and low-probability quantiles, which are of primary importance. The knowledge of confidence limits for the parameter α or β alone is not sufficient for this purpose, as both quantities are mutually correlated [7]. In the following part, a simple method for estimation of confidence limits for quantiles and values of the three-parameter distribution function is proposed; the two-parameter distribution will be addressed in another paper.

3.1 Quantile function

The lower and upper limit for the values $y(F)$ of the quantile function can generally be expressed in the form

$$y_{L,U}(F) = y(F) \pm \Phi_{L,U}(\alpha, \nu, s_y), \quad (9)$$

where Φ is a certain function of the shape of the distribution, scatter of the measured values, probability F , number of degrees of freedom and confidence level. Preliminary study of the random samples generated from a population with known parameters have shown that the experimental sample can be, as a whole, shifted aside from the parent distribution function.

Thus, the determination of confidence limits cannot be based only on the scatter of the measured values around the sample distribution function, but must also consider the variance of the position of the sample as a whole. This position (as well as its variance) can be characterised best by means of the sample mean \bar{y} . Expressing the value of the quantile function $y(F)$ as the sum of the distribution average \bar{y} and the distance of y from \bar{y} ,

$$y(F) = \bar{y} + [y(F) - \bar{y}] = \bar{y} + \delta, \quad (10)$$

one obtains the total variance of the $y(F)$ value of the quantile function as

$$s_{y(F)}^2 = s_{\bar{y}}^2 + s_{\delta}^2. \quad (11)$$

However, both quantities, \bar{y} and δ , have different distributions. The function F cannot, therefore, be expressed simply as a product of the resultant standard deviation $s_{y(F)}$ and the critical value of Student's distribution like in the case of confidence interval for the mean alone. It is better to express Φ , in analogy with Eq. (11), by the means of half-widths Δ of the confidence intervals for \bar{y} and δ :

$$\Phi = \sqrt{\Delta_{\bar{y}}^2 + \Delta_{\delta}^2}. \quad (12)$$

For larger samples, the distribution of sample means can be approximated by normal distribution, so that it can be written

$$\Delta_{\bar{y}} = t_{\alpha/2, \nu} \frac{s_y}{\sqrt{n}}, \quad (13)$$

where

$$s_y^2 = \sqrt{\frac{\sum_j (y_{mj} - \bar{y})^2}{n - 1}}, \quad (14)$$

is the sample standard deviation of y , and $t_{\alpha/2, \nu}$ is the critical value of the Student's distribution for $\nu = n - 1$ degrees of freedom; α is the probability that the true value \bar{y} will lie outside the confidence limits, i.e. $P(t > t_{\alpha, \nu}) = \alpha$. (In some tables, $t_{\alpha, \nu}$ denotes the value for which $P(|t| > t_{\alpha, \nu}) = \alpha$, and which corresponds to the twice as high probability [8]).

The determination of the width of the confidence interval for δ is more complex. With respect to the definitions (10) and (2), one can write

$$\delta(F) = a \left[\left(\ln \frac{1}{1-F} \right)^{1/b} - \left(\ln \frac{1}{1-F^*} \right)^{1/b} \right], \quad (15)$$

where F^* is the value of the distribution function corresponding to \bar{y} (Fig. 2). The variance of δ thus generally depends on the variance of the estimate of the scale parameter a and the shape parameter b (which are, moreover, mutually correlated), and on the variance of the F^* value. Exact expression of Δ_{δ} would, therefore, be very complicated. However, it is sufficient for our purposes to know only the approximate limits $\Delta_{\delta_{\min}}$ and $\Delta_{\delta_{\max}}$, between which the true value Δ_{δ} lies.

The lower limit $\Delta_{\delta(F), \min}$ for the estimate of the width of the confidence interval for δ can be obtained if one neglects the variance of the shape parameter b and of the F^* value, and considers only the random variability of the scale parameter a . This quantity has χ^2 distribution, and its lower and upper confidence limits can be determined using the expressions [1, 8]

$$a_L = a \left(\frac{2n}{\chi^2_{1-\alpha/2}(2n)} \right)^{1/\beta}, \quad a_U = a \left(\frac{2n}{\chi^2_{\alpha/2}(2n)} \right)^{1/\beta}, \quad (16)$$

where $\chi^2_{\alpha}(2n)$ is α -quantile of the χ^2 distribution for $\nu = 2n$ degrees of freedom; α is the probability that the true value of the scale parameter a will be lower than a_L or higher than a_U . If only the estimate b is known instead of the parameter β , it is better to use the following expressions:

$$a_L = a \exp\left(-\frac{z_{1-\alpha/2, n}}{b}\right), \quad a_U = a \exp\left(-\frac{z_{\alpha/2, n}}{b}\right), \quad (17)$$

where $z(\alpha, n)$ are quantiles of the random variable b/β , which are tabulated [1].

The half-width of the confidence interval for δ can be expressed (with respect to Eq. 15) as

$$\Delta_{\delta, L, \min}(F) = (a - a_L) \Psi, \quad (18a)$$

$$\Delta_{\delta, U, \min}(F) = (a_U - a) \Psi, \quad (18b)$$

where Ψ is given by the expression

$$\Psi(F, F^*, b) = \left(\ln \frac{1}{1-F} \right)^{1/b} - \left(\ln \frac{1}{1-F^*} \right)^{1/b}. \quad (19)$$

With regard to the definition (2), one can also write

$$\Psi(F, F^*, b) = \frac{y(F) - \bar{y}}{a}; \quad (20)$$

Ψ thus expresses the distance of y from the distribution mean \bar{y} , related to the scale parameter a .

Inserting (18) and (13) into (12) gives the estimate of Φ needed for the determination of the two-sided confidence limits (9) for the quantile function $y(F)$:

$$\Phi_{L,U} [y(F)] = \sqrt{t_{\alpha/2, \nu}^2 \frac{s_y^2}{n} + (a - a_{L,U})^2 \Psi^2}. \quad (21)$$

(As the expression in the parentheses is squared, it does not matter whether the term for the upper limit is written $a_U - a$ or $a - a_U$.) When calculating the lower limit Φ_L , a_U is inserted if $y < \bar{y}$, and a_L if $y > \bar{y}$. When calculating the upper limit Φ_U , a_L is inserted if $y < \bar{y}$, and a_U if $y > \bar{y}$.

The maximum possible width $\Delta\delta_{(F), \max}$ of the confidence interval for $\delta(F)$ can be estimated using the idea that the random variation of the quantile function $y(F)$ consists of the variation of the sample mean position \bar{y} , and variation of the „slope“ of the distribution function, which is caused by the fluctuations of the values a , b calculated from random samples (Fig. 2). As follows from Eq. (18), the distance between the true value of the quantile, $y_p(F)$, and the value $y(F)$ calculated from the sample values a , b is larger for larger difference between the true and calculated value a , and for larger value of the term Ψ , which also depends on b . Since there is only a low probability that both terms, a and Ψ (or b) will attain their limit values simultaneously, the maximum possible width of the confidence interval for $\delta(F)$ can again be estimated using Eq. (15), into which we now insert the limit value a_L (or a_U) of the scale parameter, and that of the limit values b_L or b_U of the shape parameter, which gives the larger value Ψ . Two-sided confidence limits for the parameter α are determined from Eq. (16) or (17); those for β can be calculated by [1]

$$b_L = b / I_{1-\alpha/2, n} \quad , \quad b_U = b / I_{\alpha/2, n} \quad , \quad (22)$$

where $I_{\alpha, \nu}$ denotes the quantiles of the random quantity b/β , which are tabulated [1]; α is the probability that the true value will lie outside the interval $< b_L; b_U >$.

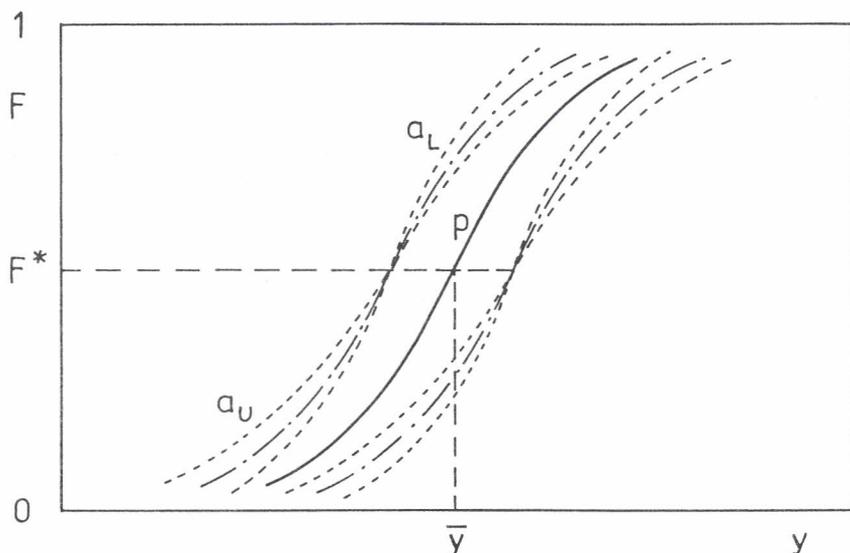


Fig. 2 Variance of sample distribution function: variance of the mean and of the slope. p - distribution function of the parent population, \bar{y} - sample mean, a - sample estimates of the scale parameter α ; L , U - lower and upper confidence limit.

Notice. A part of the variance of δ is caused by the scatter of F^* . The simulation with random samples (Section 4) has shown that this scatter, and especially that of the term $\{\ln[1/(1-F^*)]\}^{1/b}$ is relatively small and need not be taken into account.

The lower and upper limits of the confidence interval for the quantile function, $y_L(F)$ and $y_U(F)$, can again be obtained using Eqs. (9), (21) and (19). Inserting $F = 0$ gives the limits

$y_{L,U}(0) = c_{L,U}$ of the confidence interval for the threshold value γ of the three-parameter distribution.

3.2 Distribution function

The values $y_L(F)$ and $y_U(F)$ for various values of F create a confidence band, which should, with probability $1 - \alpha$, cover the true values y corresponding to the probabilities F (Fig. 3). This band can also be used for the construction of confidence intervals for the F values of distribution function, corresponding to the particular values of y . A straight line $y = \text{const}$, plotted in the coordinate system $F - y$, crosses the distribution function at the point F , and the upper and lower limit of the confidence band at the points F_U and F_L (Fig. 3).

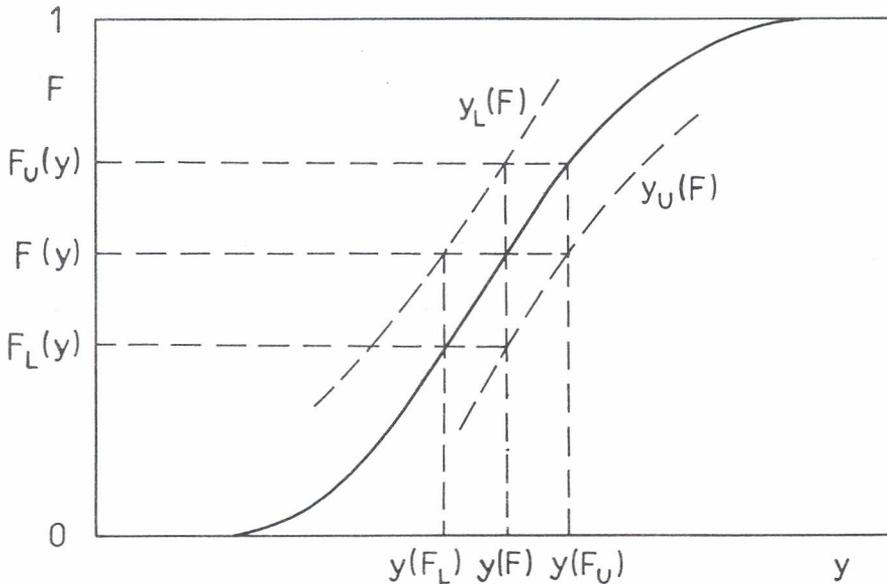


Fig. 3 Confidence limits of quantiles $y(F)$ and of distribution function $F(y)$.

L, U - lower and upper limit; $y(F) = y_L(F) = y_U(F)$.

The probability that the true value $F(y)$ will be higher than F_U or lower than F_L equals α . The point $F_U(y)$ makes the upper limit of the confidence interval for $F(y)$, and, simultaneously, the lower limit y_L of the confidence interval for the quantile corresponding to the probability F_U (horizontal line in Fig. 3). Similarly, the point $F_L(y)$ makes the upper limit y_U of the quantile for F_L . Thus,

$$y_L(F_U) = y(F) = y_U(F_L). \quad (23)$$

Expressing y_L in the left of the equations (23) as

$$y_L(F_U) = y(F_U) - \Phi_L[y(F_U)], \quad (24)$$

gives, after solving for $y(F_U)$ and making the inverse transformation, the upper limit of the probability F_U for the quantile y :

$$F_U(y) = 1 - \exp \left[- \left(\frac{y(F) + \Phi_L - c}{a} \right)^b \right]. \quad (25)$$

This expression differs from Eq. (1) for the mean value $F(y)$ by the term Φ_L determined using Eqs (21) and (19) and added to the y -value. Rigorously, the value F_U should be inserted into (19) for F , i.e. the value, which is not known yet. The calculations should, therefore, be performed in an iterative way. However, as the confidence band width varies with F only slowly, the F value can be used in (19) without significant error.

Lower limit F_L of the confidence interval for $F(y)$ is obtained if the term Φ_L in (25) is replaced by $-\Phi_U$.

The proposed approximate methods for estimation of the parameters α , β , γ of three-parameter Weibull distribution are based on several simplifying assumptions, and are thus valid only under certain limitations:

1. The minimum value of strength or time to failure cannot be lower than zero. If the computed c value is negative, the proposed distribution function is not suitable, and another distribution should be tested, e.g. two-parameter Weibull function.
2. Another kind of distribution may also be more suitable if the lower confidence limit for the threshold value γ was obtained negative, $cL = y(F=0) < 0$.
3. The threshold value cannot be higher than the lowest measured value y_1 . If cU is higher than y_1 , the upper confidence limit for $y(F)$ should not be used, especially for low quantiles.

Since in reliability assessment we need to know especially the „guaranteed minimum values“ of strength or time to failure, or the values which will not be attained only with a very low probability, it is especially the lower confidence limit $y_L(F)$ for quantiles and the upper confidence limit $F_U(y)$ for the distribution function, which are of major importance. It is thus possible to determine only these limits. One-sided confidence limits for the confidence level $1 - \alpha$ can be obtained if $\alpha/2$ in the pertinent formulae is replaced by α .

4. EXPERIMENTAL PART

The properties and suitability of the proposed methods were tested on simulated samples, generated from a population with known distribution. Pseudo-random numbers uniformly distributed in the interval (0; 1) were generated by the program Excel and transformed using Eq. (2) into the numbers having Weibull distribution with parameters $\alpha = 100$, $\beta = 2$, $\gamma = 200$. In this way, 30 samples were created with 10 values each ($n = 10$), then 30 samples with 40 values each, and 30 samples with 100 values. Always, the j -th sample with 10 values was contained in the j -th sample with 40 values, which, again, created a part of the j -th sample with 100 values. For all 90 random samples, the coefficients a , b , c of the three-parameter distribution were determined, using:

- a) method of moments, using the tables in [1],
- b) method of least squares, with the calculation of cumulative probabilities F_j according to Hazen, Eq. (6a), and with the minimisation of the residual standard deviation s_{res} defined by Eq. (7),
- c) method of least squares like b), but with the cumulative probabilities determined according to Gumbel; Eq. (6c),

- d) method of least squares, with the cumulative probabilities calculated according to Eq. (6a), and with the maximisation of the correlation coefficient r .

The least squares method was also used for the determination of the constants a , b of two-parameter distribution (assumed $c = 0$) for three groups of samples (with $n = 10, 40$ and 100). Confidence intervals (for $1 - \alpha = 0,9$) were determined for the quantiles of the three-parameter empirical distributions, and also for the parameters α , β of these distributions, as well as of the parent distribution, using tables in [1].

All files with the input y_i , F_j data, calculated distribution characteristics and graphs are archived at the KDMPS in printed form as the appendix to the '97 Report of the grant project GA ČR 103/97/0139, and also stored on 3,5" discs as Excel files. The individual files are denoted by alpha-numerical code; e.g. W10GS09: the letter W means Weibull distribution, the following number (10, 40 or 100) denotes the sample size, the following letter (G or H) tells whether the cumulative probabilities F_j were determined according to Gumbel or Hazen, the following letter denotes the optimisation criterion (S - minimum of the residual standard deviation, R - maximum of the correlation coefficient r , Z - two-parameter distribution with zero threshold value, $c = 0$). The sample number is given by the last two numbers (01 through 30). The data necessary for the method of moments and the pertinent results are contained in the files for three-parameter distribution.

Fig. 1 shows a random sample and its approximation by the two- and three- parameter distributions. Figs. 4 - 6 show examples of three-parameter distributions. The solid curve corresponds to the parent function (with parameters α , β , γ), dashed curve is the empirical distribution function with the parameters determined by the method of moments, and dash-and-dot curve belongs to the distribution function with the parameters determined by the least squares method. The confidence limits (dotted curves) were determined by Eq. (9) for the distribution function obtained by the moment method. The confidence level in all cases was $1 - \alpha = 0,90$. A brief overview of all results is given in Table 1.

An extremely important lesson from this simulation of 90 samples is that relatively often the samples were significantly shifted aside from the parent distribution function (Fig. 4). Also the slope of the sample distribution sometimes markedly differed from that of the parent distribution function, especially for small samples (Fig. 5). With increasing sample size n , the scatter of the measured values around the empirical distribution function got smaller, and their distribution in the $F(y)$ diagram came closer to the parent function (Figs. 5, 6).

The width of the confidence bands for three-parameter distribution were smallest for $y = \bar{y}$ (equal to the width of the confidence interval for the mean), and gradually grew for both y growing and decreasing. Generally, the band width decreased with growing sample size (in correspondence with the critical values $t_{\alpha, \nu}$ and $z_{\alpha, \nu}$, as well as the standard deviation of the mean, which is indirectly proportional to the square root of n ; see Eqs. (21), (13) and (16).

The experience obtained from the approximations of the quantile (or distribution) functions using the five methods can be divided according to the kind of the distribution.

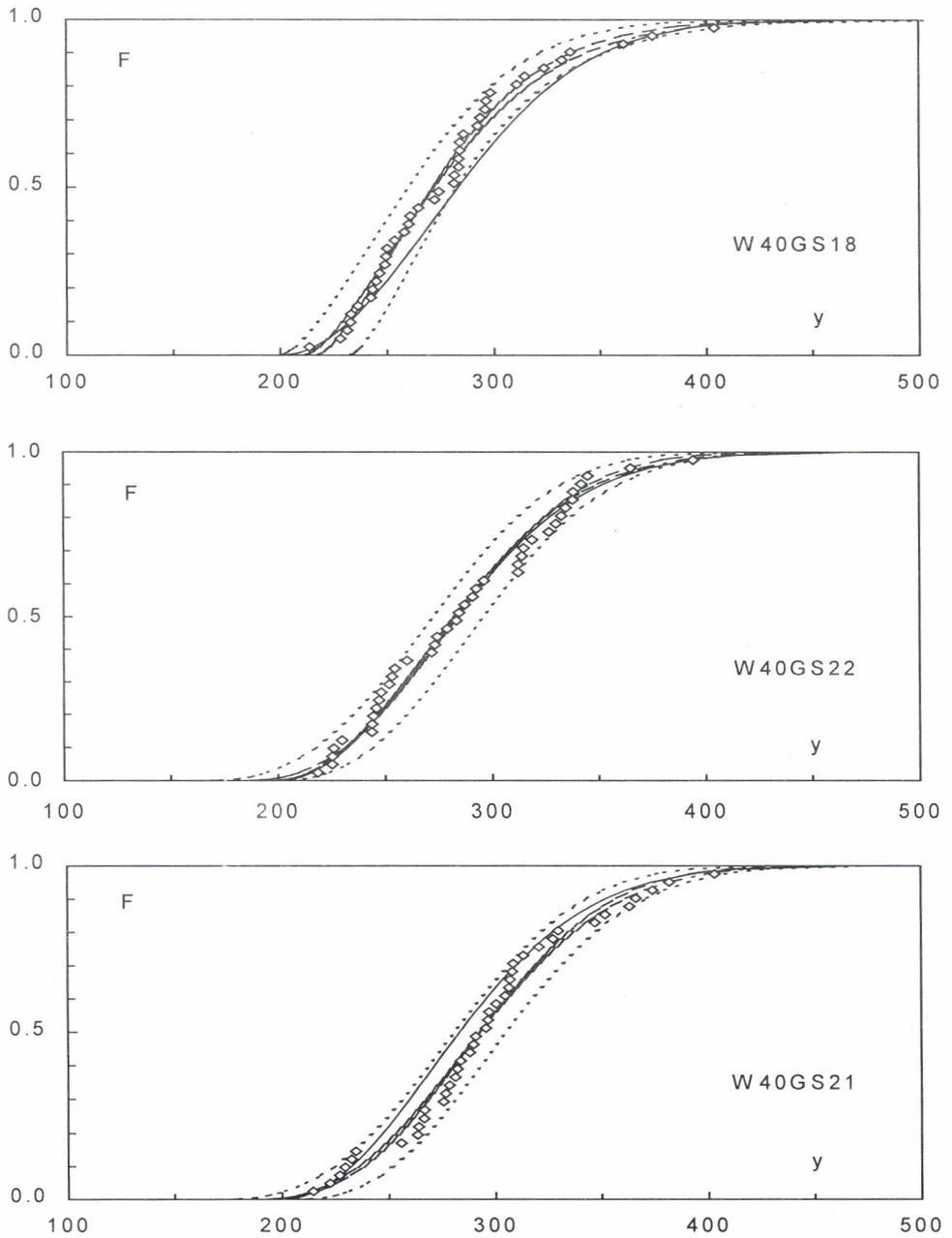


Fig. 4 Three random samples of size $n = 40$, approximated by three-parameter Weibull distribution.

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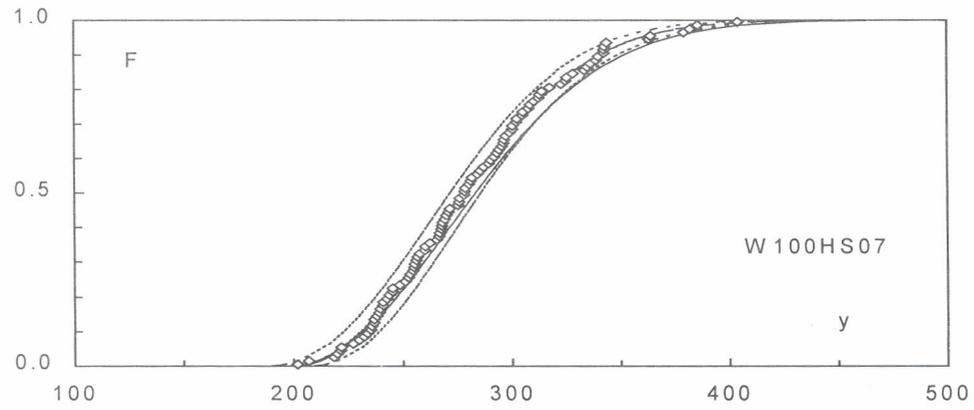
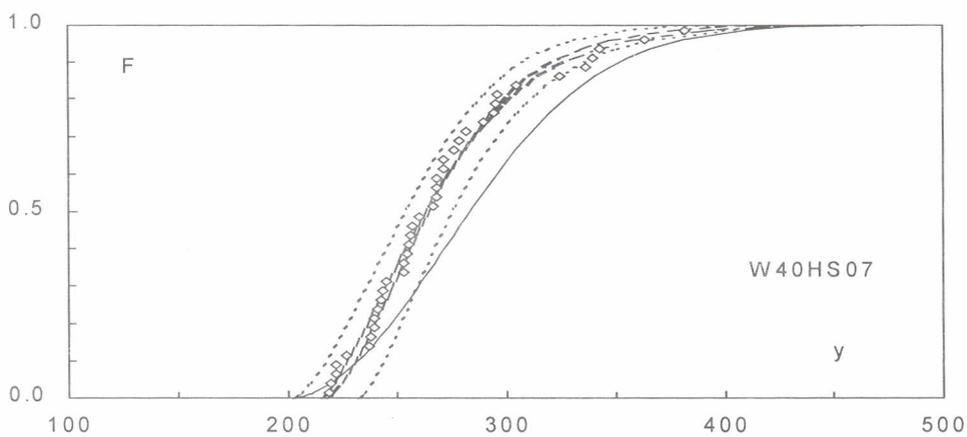
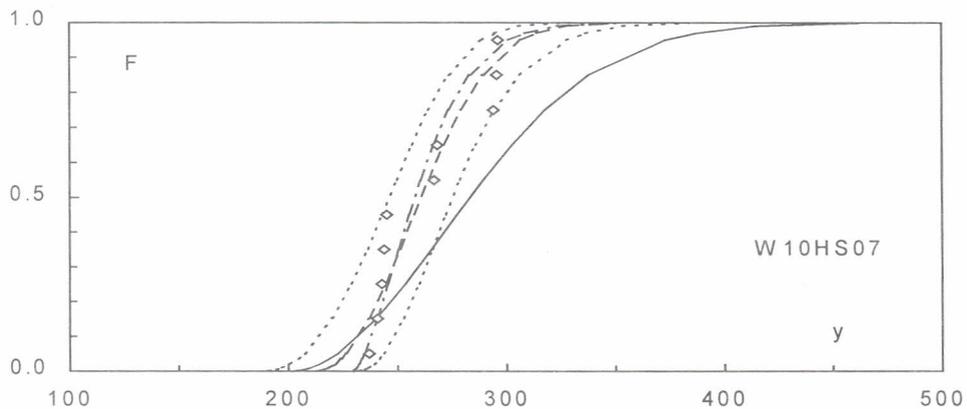


Fig. 5 Three random samples of size $n = 10 - 40 - 100$, approximated by three-parameter Weibull distribution. (Larger sample always contain all values from the smaller one.)

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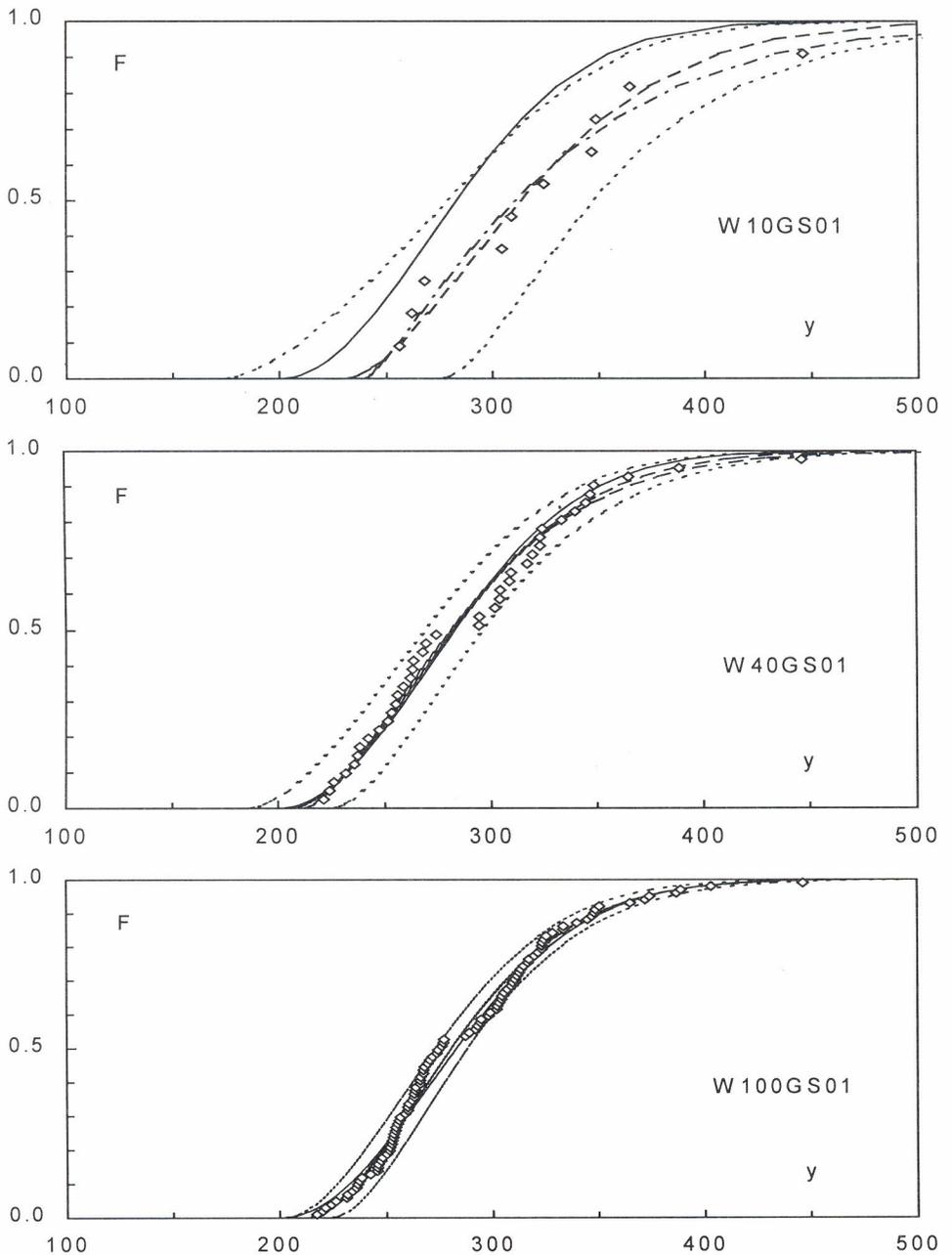


Fig. 6 Three random samples of size $n = 10 - 40 - 100$, approximated by three-parameter Weibull distribution. (Larger sample always contain all values from the smaller one.)

_____ parent distribution function, - - - - - approximation with parameters determined by the method of moments, (- . - . - least squares method),
 confidence limits for $1 - \alpha = 0,90$.

Method	c_{min}	c_{max}	$n(c>\gamma)$	$c_{L,min}$	$c_{L,max}$	$n(c_L>\gamma)$	$c_{U,min}$	$c_{U,max}$	$n(c_U>\gamma)$
W10MM	-85.1	238.8	11	-124.0	203.0	1	-47.9	276.2	10
W10HS	0.0	252.8	15	-39.6	207.4	2	34.1	289.0	10
W10HR	0.0	252.1	17	-39.6	211.7	3	34.1	299.7	8
W10GS	0.0	252.8	14	-47.7	200.9	1	37.4	288.1	11
W40MM	147.2	226.7	16	132.8	211.8	7	161.1	248.8	8
W40HS	168.5	223.8	18	154.2	208.7	3	182.3	244.6	3
W40HR	180.4	228.1	18	161.7	212.3	4	196.0	242.5	1
W40GS	154.1	222.0	16	139.3	206.1	2	168.3	245.0	4
W100MM	168.5	220.5	15	159.6	207.7	6	177.2	232.5	10
W100HS	180.3	216.3	17	170.2	205.6	2	190.1	226.4	2
W100HR	189.5	217.8	15	178.3	206.9	2	199.5	228.1	1
W100GS	176.1	215.5	17	165.8	204.6	2	186.1	225.8	3

Method	a_{min}	a_{max}	$n(a>\alpha)$	b_{min}	b_{max}	$n(b>\beta)$	$y_{1,min}$	$y_{1,max}$	n
W10MM	52.4	424.5	19	1.15	10.25	15	202.1	256.2	10
W10HS	34.4	338.6	16	0.89	8.54	14	202.1	256.2	10
W10HR	26.6	338.6	13	0.75	8.54	16	202.1	256.2	10
W10GS	36.2	340.8	16	0.74	7.02	14	202.1	256.2	10
W40MM	48.1	157.3	15	1.05	4.15	17	202.1	233.8	40
W40HS	61.9	133.7	12	1.30	3.60	13	202.1	233.8	40
W40HR	62.4	126.0	11	1.36	2.90	13	202.1	233.8	40
W40GS	63.4	148.8	14	1.22	3.77	12	202.1	233.8	40
W100MM	59.7	136.3	15	1.15	3.29	15	200.7	222.2	100
W100HS	78.0	126.8	12	1.61	2.87	15	200.7	222.2	100
W100HR	76.4	123.2	13	1.55	2.58	14	200.7	222.2	100
W100GS	79.6	131.8	14	1.59	2.87	13	200.7	222.2	100

Tab. 1 Scatter in the simulated samples; influence of the method and sample size.

MM - distribution parameters determined by the method of moments, H, G - parameters determined by the least squares method, with cumulative probabilities calculated according to Hazen (H) and Gumbel (G); S, R - optimisation criterion: S - minimum residual standard deviation, R - maximum correlation coefficient.

min, max - minimum and maximum value of 30 samples, L, U - lower and upper confidence limit ($1-\alpha = 0,90$), $n(a>\alpha)$, $n(b>\beta)$, $n(c>\gamma)$ - number of cases (in 30 samples) with the parameter estimates a , b , c exceeding the true parameters $\alpha = 100$, $\beta = 2$, $\gamma = 200$.

4.1 Two-parameter distribution

This distribution has zero threshold value ($c = \gamma = 0$), and also the values of quantiles $y(F)$ for very low probabilities F were always smaller than the quantiles $y_p(F)$ of the parent (three-parameter) distribution (Fig. 1). However, for higher (but still relatively low) values of F , the quantiles calculated using the sample distribution function were often higher than the quantiles of the parent distribution (Fig. 1). With $n = 10$, for example, $y(F=0,01)$ was larger than $y_p(F = 0,01)$ in one case of 20, $y(F=0,03)$ was larger than y_p in three cases of 20, and $y(F=0,1)$ was larger than y_p in twelve cases of 20. With larger samples, the quantiles of the

empirical distribution functions were lower than the quantiles of the parent function up to higher values of F , but later they exceeded the y_p value more often. With $n = 40$, for example, $y(F=0,03)$ was larger than $y_p(F = 0,03)$ in one case of 20, but for $F = 0,1$ y was larger than y_p in 15 cases of 20.

It is obvious from these results that even the two-parameter distribution (especially if used without checking the quality of the fit) need not guarantee safe conclusions, especially for higher quantiles.

4.2 Three-parameter distribution

The main conclusions from the simulation are as follows:

1. The differences between the distribution functions determined for a certain sample by the four methods a) - d) were, in general, smaller than the differences between the individual samples. Often, the distribution function obtained by the least squares method coincided almost with the function obtained by the method of moments, though there were also significant differences in several cases (e.g. sample W10HS07 in Fig. 5 and W10GS01 in Fig. 6).
2. The empirical coefficients a , b , c of the three-parameter distributions were in approximately one half of all cases (regardless the method of their determination) smaller, and in one half larger than the parameters α , β , γ of the parent population. In several cases, the constants a , b were significantly out of the confidence limits constructed for α , β . The variation of the constants a , b , c was generally larger for smaller samples.
3. The calculated constant c was in about one half of all cases lower and in one half higher than the true threshold value γ of the parent distribution, and its variation was large. The distribution of the generated values was such that for two of the smallest samples ($n = 10$) c was obtained negative using both the method of moments and least squares. On the other hand, the maximum value of c significantly exceeded the threshold value $\gamma = 200$. In the group of 30 samples, $c_{\max} = 220,5$ for the sample size $n = 100$, $c_{\max} = 228,1$ for $n = 40$, and $c_{\max} = 252,8$ for $n = 10$; see Table 1. This means that reliability assessment or dimensioning for a very low (acceptable) probability of failure cannot be based only on the distribution function (or the threshold value c) determined by the least squares method or by the method of moments (and, perhaps, by the method of maximum likelihood, as well), but it is necessary to use the confidence interval for these estimates also.
4. Two-sided confidence band, constructed, using Eqs. (9), (21) and (19), for the minimum width $\Delta_{\delta(F), \min}$, covered (at confidence level $1 - \alpha = 0,9$) in most cases the parent function as well as the experimental values y_j (Figs. 4 - 6). Sometimes, however, the distribution function left the band in a certain region (sample W40GS18 in Fig. 4, samples W10HS07 and W40HS07 in Fig. 5, and sample W10GS01 in Fig. 6, for example). In several cases the calculated lower limit c_L was higher than the true threshold value γ . Such kind of error can be dangerous when the „guaranteed“ values should be determined, and was, therefore, studied in more detail.

The safest results were obtained by the variant c) of the least squares method, i.e. that determining the cumulative probabilities according to Gumbel, Eq. (3c), and minimising the

residual standard deviation s_{res} defined by Eq. (7). With sample size $n = 10$, c_L was higher than γ in one case of 30. With $n = 40$, c_L was higher than γ in two cases of 30, and the same result was obtained for $n = 100$ (Table 1). These five one-side exceedings in 90 cases correspond approximately to the probability 5%, which can be expected for two-sided interval at confidence level $1 - \alpha = 90\%$. When $1 - \alpha$ was raised to 0,96, c_L was higher than γ only in one case of 90, which again corresponds (safely) to the statistical definition of the interval. The variant c) gave also the lowest exceedings of the threshold value: with $1 - \alpha = 0,90$, the highest value $c_{L, max}$ obtained for $n = 10$ was 200,9, for $n = 100$ it was 204,6, and for 40 it was 206,0 (and dropped to 201,9 for $1 - \alpha = 0,96$).

The results of other methods were slightly worse. With $1 - \alpha = 0,9$, the variant b) gave $c_L > \gamma$ in 7 cases of 90, the variant d) gave $c_L > \gamma$ in 9 cases of 90, and the method of moments gave $c_L > \gamma$ in 14 cases of 90. Also the maximum values of c_L were higher (Table 1). The thorough evaluation of the individual methods, however, would need a larger number of experiments.

The upper limit c_U of the confidence interval was mostly (not always) higher than the threshold value γ (Tab. 1). For more than one half of all cases c_U was even higher than the lowest measured value y_1 , which contradicts to the reality. That means that Eqs. (9) and (21) for the determination of the upper limit of $y(F)$ are not suitable for low probabilities F . Fortunately, it is the lower confidence limit (i.e. y_L or c_L), which is important for the determination of the guaranteed minimum values, (and, analogously, the upper limit F_U for the probability F).

Let us now consider the influence of individual factors on the width of the confidence band for distribution function. This width is minimum at $y = \bar{y}$ (equal to the width of confidence band for the mean value), and grows continuously for y increasing as well as decreasing. At the threshold value $y = c$ (i. e. for $F = 0$) the width was about 40 - 70 percent larger than that for \bar{y} , the increase being larger for smaller samples (Fig. 6). Similar widths were again attained for F as high as 0,9 - 0,95. For higher values F the width grew faster, and at $F = 0,999$ was about twice or three-times that for \bar{y} . These numbers correspond to the case, where only the variance of the average value \bar{y} and of the scale parameter a were considered (i.e., $\Delta_{\delta(F), min}$ was used for the determination of Φ). Some role is also played by the variance of the distribution shape parameter b and of the value F^* in the distribution function, corresponding to the average value \bar{y} . The influence of F^* is small, partly because it appears in the expression (15) for b only in the term $\{ln[1/(1-F^*)]\}^{1/b}$. The coefficient of variation of this term was 3,7% for 30 samples of size $n = 10$, 1% for 30 samples with $n = 40$, and less than half percent for 30 samples with $n = 100$. Thus, the influence of the variance of F^* can be neglected. The influence of the variance of b was, except for high probabilities F , also small. At the threshold value c (i. e. for $F = 0$), the confidence intervals considering the upper limit b_U in Eq. (19) instead of b were wider by about 2% for $n = 10$ [i.e. $\Phi_{max}(F = 0) \approx 1,02 \Phi_{min}(F = 0)$]. For $n = 40$ they were wider by 1%, and for $n = 100$ they were wider by 0,8%. (For $F > 1\%$, however, these intervals were narrower than those constructed without considering the scatter in b , while slightly wider were the intervals constructed using the lower limit b_L in Eq. 19). The difference between the Φ values, determined for $\Delta_{\delta, min}$ and $\Delta_{\delta, max}$ was smaller than 2% for F up to $F = 0,8$ and began to grow faster only for $F > 0,9$, especially for small samples. The widths of confidence intervals for $F = 0,999$ (and for $\Delta_{\delta, max}$ determined using the lower limit b_L)

were for $n = 100$ larger by about 20%, and for $n = 40$ they were larger by 50% compared to the case not considering the variance of b . For $n = 10$, the increase of width was even faster, and, for very high values of F , the solution lost stability; for example, the lower limit of confidence interval for $F > 0,999$ was even lower than that for $F = 0,75$. With respect to this ambiguous character, and especially due to the small influence of the scatter of the parameter b for most values of F and for larger samples, it is thus preferable to determine confidence intervals by the simpler method (i. e. for $\Delta_{\delta(F),\min}$), inserting the sample estimate b of the shape parameter into (19). The reliability of the estimate can be increased by slightly increasing the confidence level $1 - \alpha$.

5. SUMMARY

Four variants of determining the parameters of Weibull distribution by the least squares method were proposed and tested, and also a method for obtaining confidence limits for the quantile and distribution functions of the three-parameter distribution. The individual methods for parameter estimation differed in ordering cumulative probabilities F_j to the measured values y_j (see Eqs. 6a-c in Section 2.1), and by the criterion for finding the optimum estimates of the parameters (minimum residual standard deviation of the measured values from the sample distribution function, or the maximum value of the correlation coefficient, see Section 2.2).

The proposed method for determining the confidence intervals for quantile function is based on decomposing the total variation of the $y(F)$ value into the variation of the sample average and the variation of the slope of the sample distribution function. With respect to the complex relationships between all these quantities, simpler expressions were developed for the determination of the minimum and maximum possible width of confidence intervals (considering only the variation of \bar{y} and a in the former case, and of \bar{y} , a and b in the latter; see Section 3.1). The confidence limits for distribution function $F(y)$ can be obtained by a simple transformation of the expressions for confidence limits of the quantile function (Section 3.2).

The differences between the three-parameter distribution functions, obtained for a particular sample by the methods a) - d) as well as by the method of moments were, in general, smaller than the differences between the individual samples. Also, the two-sided 90%-confidence intervals, constructed using Eqs. (9), (21) and (19), covered in most cases the substantial part of the parent distribution function and of the measured values, regardless the method for parameter determination as well as the kind of confidence interval (i.e. corresponding to the minimum or maximum possible width of Δ).

A more comprehensive evaluation of the individual methods would need more extensive testing. Nevertheless, in the accomplished tests the safest results were obtained by the variant c), i. e. with determining the cumulative probabilities by the Gumbel's expression $F_j = j/(n+1)$ and minimising the residual standard deviation around the untransformed distribution function. The number of samples for which the lower confidence limit c_L was higher than the true threshold value γ of the parent distribution was approximately the same for all groups of samples (i. e. independent of the sample size) and corresponded to the chosen confidence level. It has also been proven that (except for very high values F) the scatter of the shape parameter b is small compared to the scatter of the sample average \bar{y} and of the scale parameter a . Thus the confidence limits can be estimated by inserting the

sample estimate for b into Eq. (19), without considering its scatter. Sufficient reliability of the estimate can be obtained by choosing a suitable confidence level $1 - \alpha$.

The variation of experimental values was pronounced especially for small samples, where also the confidence interval were widest. In some cases the calculated threshold value c was much lower than the true value γ . In such cases, another distribution should be chosen. Always it is necessary to check the fit.

With increasing sample size n , the distribution of experimental values approached to the parent distribution, and the width of the confidence band decreased. The accuracy of the estimate can thus be increased by increasing n . For relatively narrow intervals at high confidence level, large sample size is necessary, with $n = 100$ or more. When deciding about the extent of measurement and the confidence level, it is always necessary to compare the testing costs with those due to over-dimensioning the structure (due to wide confidence intervals), and the costs caused by the eventual failure.

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Resumé

POUŽITÍ WEIBULLOVA ROZDĚLENÍ PŘI POSUZOVÁNÍ SPOLEHLIVOSTI

Jaroslav MENČÍK

Je popsán způsob odhadu parametrů Weibullova trojparametrického rozdělení metodou nejmenších čtverců, a stanovení konfidenčních mezí pro hodnoty kvantilů a distribuční funkce. Testování pomocí náhodných výběrů, generovaných počítačem ze základního souboru se známým rozdělením, ukázalo, že výběrový soubor se při malém rozsahu poměrně často lišil od základního souboru polohou nebo sklonem, a výběrové charakteristiky se pohybovaly kolem skutečných parametrů v širokých mezích. Při výpočtu kvantilů nebo hodnot distribuční funkce je proto nutno určovat vždy konfidenční intervaly. Navržený způsob jejich stanovení, vycházející z rozptylu polohy středu a sklonu výběrového rozdělení, zajišťuje dostatečnou spolehlivost odhadu zejména dolní konfidenční meze pro kvantily a horní meze pro hodnoty distribuční funkce. K dispozici je i česká verze článku.

Summary

THE USE OF WEIBULL DISTRIBUTION FOR RELIABILITY ASSESSMENT

Jaroslav MENČÍK

A method is proposed for estimation of the parameters of Weibull three-parameter distribution (based on the least squares method), and a method for the determination of confidence limits for the values of quantile and distribution functions. Testing of 90 random samples, computer-generated from a distribution with known parameters, has shown that the distribution of small samples often differed from the parent distribution by the position or slope. Thus, when determining quantiles or values of the distribution function, confidence intervals must also be determined. The proposed method for their obtaining (based on the decomposition of the total variance into the variance of the mean and slope of the sample distribution function) ensures a sufficient reliability especially for the estimates of the lower confidence limit for quantile function and upper limit for the values of distribution function.

Zusammenfassung

DIE VERWENDUNG VON WEIBULLVERTEILUNG FÜR ZUVERLÄSSIGKEITSBEURTEILUNG

Jaroslav MENČÍK

Der Aufsatz beschreibt eine Methode für Abschätzung der Parametern von Weibull Dreiparameterverteilung mit der Methode der kleinsten Quadrate, und auch eine Methode für die Bestimmung von Konfidenzgrenzen für Quantile und Werte der Distributionsfunktion. Die Methoden wurden getestet mittels 90 Zufallsproben generiert aus einer Verteilung mit bekannten Parametern. Es hat sich gezeigt, daß besonders kleine Proben unterschieden sich oft von der Grundverteilung entweder durch die Lage oder Neigung der Distributionsfunktion. Die entworfene Methode für die Konfidenzgrenzenbestimmung, die die Streuung beider dieser Größen berücksichtigt, gibt genügende Zuverlässigkeit besonders für die untere Grenze für Quantile und die obere Grenze für die Werte der Distributionsfunktion.

Jaroslav Menčík:

The use of Weibull distribution for reliability assessment