NOTES ON SELF-SIMILARITY

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Abstract: *The properties of self-similar sets are discussed and a brief historical survey of ideas related to the notion of self-similarity is presented. An example of a self-similar set is introduced and its Hausdorff and box dimensions are computed.*

Keywords*: fractals, self-similarity, von Koch curve, Hausdorff dimension*

1. Introduction

Let (X,d) be a metric space and $d(x,y)$ the distance between x and y in (X,d). A map $S:(X,d)\to(X,d)$ is called a *similarity* with ratio c if there exists a number $c>0$ such that $d(S(x),S(y))=cd(x,y)$ for all $x,y\in(X,d)$. Often (X,d) is the space $\mathbf{R}^n$ with the Euclidean metric, i.e. $d(x,y)=\sqrt{\sum_{i=1}^{n}|x_i-y_i|^2}$. Geometrically seen, similarity transformations include a homothety, an isometry and their compositions. Two sets are similar, if one is the image of the other under a similarity transformation. A set E is said to be *self-similar*, if it can be expressed as a union of m similar images of itself, that is,

$$E=\bigcup_{k=1}^{m}S_k(E). \tag{1}$$

A self-similar set defined by (1) may be seen as the *invariant set* or the *attractor* of an *iterated function system* (IFS), where all functions $S_1,\ldots,S_m$ of the IFS are similarities with ratios $c_1,\ldots,c_m$. If $c_k<1$ for all k, then all of these transformations are contractions (the corresponding IFS is sometimes called *hyperbolic*). Theory of IFS guarantees that if the space (X,d) is complete and all $S_1,\ldots,S_m$ are contractions, there exists a unique nonempty compact set defined by (1).

The construction of an invariant set E may start with any nonempty compact set F_0. For every $n>0$ we take $F_{n+1}=\bigcup_{k=1}^{m}S_k(F_n)$. The sequence $\{F_n\}$ converges (is “attracting” to) the set E (in the Hausdorff metric).

A concept related to self-similarity is self-affinity. A *self-affine* set is defined in the same way as a self-similar one with affine transformations instead of similarities. An affine transformation consists of a linear transformation and a translation and may contract with different ratios in different directions. Self-similarity is thus a particular case of self-affinity.

Some form of self-similarity has been recognized as a typical property of objects introduced as counterexamples or exceptions to proposed general rules in mathematical analysis and considered now as early examples of fractals. The word “fractal” appeared for the first time in B. Mandelbrot's book “Les objets fractals: forme, hasard et dimension”, which was published in 1975; its extended English version appeared two years later [7]. Although Mandelbrot is not an inventor of fractals, his contribution consists in revealing common features behind these objects and shapes that can be found in nature. One of these features was that of self-similarity.

Mandelbrot’s ideas brought wide public attention when his famous book “The Fractal Geometry of Nature” was published [8]. Fractal theory has been used to model various real-world phenomena, from distribution of galaxies to applications in the world of finance, including stock market analysis. These models employ rather statistical than strict self-similarity, which is typical for many fractals in mathematics.

2. Development of the notion of self-similarity and related concepts

A standard example of a self-similar set is the Cantor ternary set, which is now regarded as one of the earliest fractals. Another well-known self-similar set is the von Koch curve, introduced in 1904 as an example of a continuous curve having no tangents, constructible by means of elementary geometry. It was E. Cesàro, who first noticed the importance of self-similarity of the von Koch curve. In his paper [2], which appeared only one year after the original von Koch's memoir, he writes: “If it [the curve] was gifted with life, it would not be possible to destroy it altogether, for it will be reborn ceaselessly from the depths of its triangles, just like life in the universe.”

The general study of curves consisting of parts similar to the whole, including that of von Koch, was published in 1938 by P. Lévy [6]. Lévy notices that the part of his treatise related to the von Koch curve is based on results presented as early as 1908. Lévy introduces his own example of a plane curve constructed recursively in the same manner as the curve of von Koch. The limiting curve is now known as the Lévy C curve or the Lévy dragon.

The basic mathematical tool to describe fractals is the fractal dimension. Its definition goes back to F. Hausdorff, who defined the dimension which can take non-integer values in 1918. Another classical definition is that of Minkowski-Bouligand, mostly referred to as the box or box-counting dimension. Evaluation of the Hausdorff dimension, which is usually difficult, turns out to be quite easy in case of (strictly) self-similar sets. For the definition of the Hausdorff and box dimensions, see e.g. [3].

The problem of the Hausdorff dimension of self-similar sets was studied by P. A. P. Moran in his paper [9]. Let $S_1, \ldots, S_m$ be a collection of similarities with ratios $c_1, \ldots, c_m$ and let $\sum_{k=1}^{m} c_k^D = 1$ for some $D > 0$. Then the number D equals the Hausdorff dimension if some additional “separation” condition, called the open set condition, is satisfied.

The collection of transformations $S_1, \ldots, S_m$ satisfies the *open set condition* if there exists a nonempty bounded open set U such that $S_i(U) \subset U$ for $i = 1, \ldots, m$, and $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$.

The general theory of self-similar sets was developed by J. E. Hutchinson in 1981 [5]. Self-similar sets were also studied in general framework independently by M. Hata [4]. The concept of IFS may be found in the paper [9] by Moran (if not earlier). It was developed thoroughly by M. Barnsley in his influential book “Fractals Everywhere” [1].

3. An example of a self-similar set

Let (X, d) be a set $\mathbf{C}$ of complex numbers equipped with the usual metric, i.e. $d(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in \mathbf{C}$. For $k = 1, \ldots, m$, $m \geq 2$ we define

$$S_k(z) = w^{k-1}(az + 1), \tag{2}$$

where $0 < a < 1$ and $w = e^{\frac{2\pi i}{m}}$. Every S_k is thus a contraction with the ratio $|aw^{k-1}| = a$ and the collection of transformations $S_1, \ldots, S_m$ determines a self-similar set E (see Fig. 1).

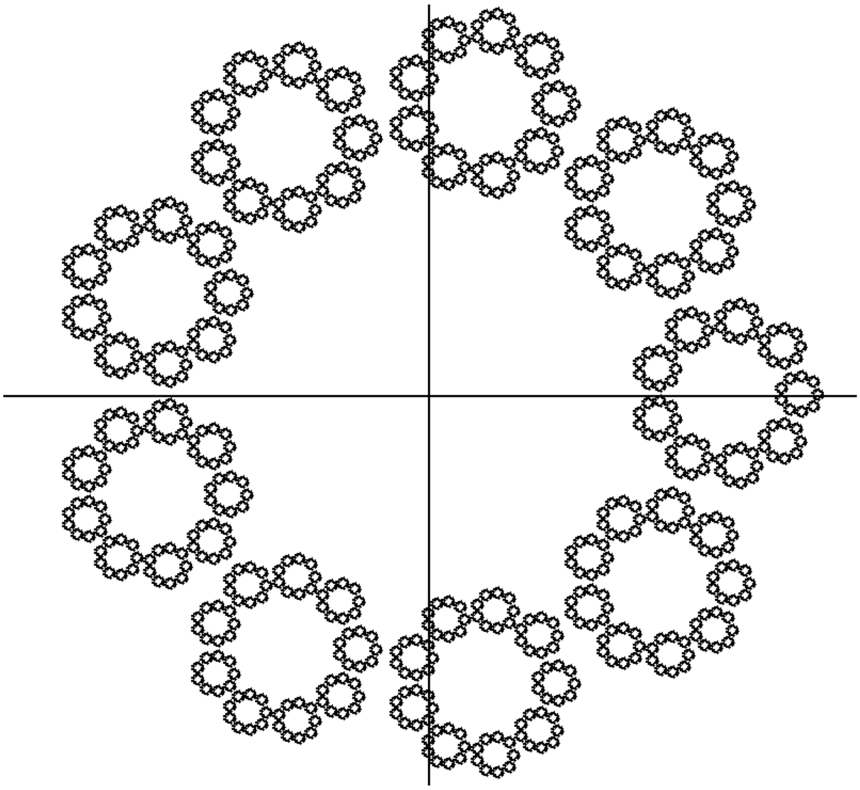

Fig. 1: The self-similar set defined by transformations (2) with $m = 9$ and $a = 1/4$.

According to the result of Williams (see e.g. [10], p. 25), the set E is connected if $a \geq l$, where $l = 1/[2(1 + \sum_{1 \leq j < m/4} \cos \frac{2j\pi}{m})]$. If $a \leq l$, then the set E satisfies the open set condition, if we take the interior of the regular polygon with vertices $\frac{w^{k-1}}{1-a}$, $k = 1, \ldots, m$ for the set U. The Hausdorff dimension D of the set E is then the unique solution of the equation $ma^D = 1$, i. e. $D = -\frac{\log m}{\log a}$.

The box dimension is equal to the same value D, which can be easily verified by the following calculation. Let B_1 be a cover of E consisting of m balls of diameter $d_1 = 2a/(1-a)$, centered at the points w^{k-1}, $k = 1, \ldots, m$ (i. e. images of the point 0 under transformations $S_1, \ldots, S_m$), B_2 a cover of E consisting of m^2 balls of diameter $d_2 = 2a^2/(1-a)$, centered at the images of 0 under transformations $S_1 \circ S_1, S_1 \circ S_2, \ldots,$ $S_m \circ S_m$, etc. B_n is thus a cover which consists of m^n balls of diameter $d_n = 2a^n/(1-a)$.

Since every member of B_n intersects *E*, the box counting dimension is exactly equal to $\lim_{n\to\infty} \frac{\log m^n}{\log(1/d_n)} = D$.

Note that for $a = l$ and $m \geq 5$ we obtain a connected set with no overlaps, which consists of infinitely many similar arcs reminding the von Koch curve. Construction of an arc *C* for $m = 9$ with the line segment [0,1] as initial set (for simplicity) is depicted in Fig. 2.

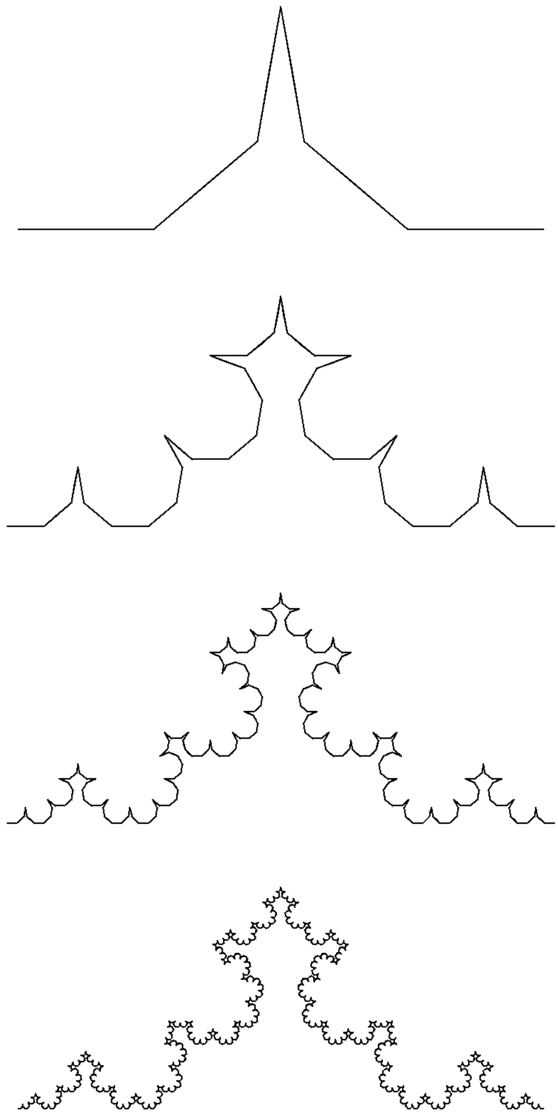

Fig. 2: Approximating polygons for the Koch-like curve.

The arc *C* is self-similar, since it can be taken as the invariant set of a collection of contractions

$$S_1(z) = az + b_1, \quad S_2(z) = aw\, z + b_2, \quad S_3(z) = aw^2\, z + b_3, \quad (3)$$

$$S_4(z) = a\overline{w^2} z + b_4, \quad S_5(z) = a\overline{w}\, z + b_5, \quad S_6(z) = a\, z + b_6$$

Here b_1 represents the point 0 and $b_j = S_{j-1}(1)$ for $j > 1$. Since C consists of six similar copies of itself scaled by the factor a and the set C satisfies the open set condition, its Hausdorff dimension is given as the solution of the equation $6a^D = 1$ and equals $D = -\frac{\log 6}{\log a} \approx 1{,}32167$.

The set C is a Jordan curve (a continuous curve without multiple points); the continuity follows from the fact that the sequence of approximating polygons in Fig. 2 converges uniformly to C. In general, these curves can be constructed by $l = 2(1 + p)$ transformations, where p is the largest integer less than m/4.

4. Conclusion

Methods of fractal analysis are widely used in many areas of science, including financial mathematics, geology, biology, computer science and others. Fractal characteristics of various real world phenomena may provide useful additional information for understanding the underlying nature of the observed processes. Functions whose graphs are fractal sets are sometimes more adequate to interpolate real data than smooth functions. Some form of self-similarity is typical for many fractal sets. Investigation of self-similar sets has been therefore important not only for purely mathematical reasons, but also for their usefulness in applications.

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